DISTRIBUTION AND NON-VANISHING OF SPECIAL VALUES OF L-SERIES ATTACHED TO ERDŐS FUNCTIONS

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ABSTRACT. In a written correspondence with A. Livingston, Erdős conjectured that for any arithmetical function f, periodic with period q, taking values in $\{-1, 1\}$ when $q \nmid n$ and $f(n) = 0$ when $q \mid n$, the series $\sum_{n=1}^{\infty} f(n)/n$ does not vanish. This conjecture is still open in the case $q \equiv 1 \bmod 4$ or when $2\phi(q) + 1 \leq q$. In this paper, we obtain the characteristic function of the limiting distribution of $L(k, f)$ for any positive integer k and Erdős function f with the same parity as k. Moreover, we show that the Erdős conjecture is true with "probability" one.

Distribution of values of L-series; Non-vanishing of values of L-series; Moments of values of L-series; Erdős's conjecture.

1. INTRODUCTION

Inspired by Dirichlet's theorem that $L(1, \chi) \neq 0$ for a non-principal Dirichlet character χ, Sarvadaman Chowla [8] initiated the study of non-vanishing of the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

for any periodic arithmetical function f whenever the above series converges. Since then, this question has been extensively studied by many authors in a variety of settings (see, for example [26], [18], [17], [7] etc.). Most of the study is concentrated around the case when f is supported on the coprime residue classes modulo q. However, very little is known otherwise.

Possibly the simplest example of an investigation in this scenario is the following conjecture made by Erdős in a written correspondence with A. Livingston [16].

Conjecture 1. *Let q be a positive integer. Let f be an arithmetical function, periodic with period q such that*

$$f(n) = \begin{cases} \pm 1 & \text{if } q \nmid n, \\ 0 & \text{if } q \mid n. \end{cases}$$

Then the series $\sum_{n=1}^{\infty} f(n)/n \neq 0$, whenever it converges.

For any periodic function f, one can define the L-series

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

which converges absolutely for $\Re(s) > 1$. Using the theory of the Hurwitz zeta-function, this series can be analytically continued to the entire complex plane except for a simple pole at

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$s = 1$ with residue $\frac{1}{q} \sum_{a=1}^{q} f(a)$. Thus, we see that the series $L(1, f) = \sum_{n=1}^{\infty} f(n)/n$ converges if and only if $\sum_{a=1}^{q} f(a) = 0$. Hence, for the sake of brevity, we say that a rational valued function f on the integers, periodic with period q is an Erdős function mod q if $f(n) \in \{-1, 1\}$ when $q \nmid n$ and $f(n) = 0$ otherwise and $\sum_{a=1}^{q} f(a) = 0$.

Erdős functions may be viewed as non-multiplicative analogues of quadratic Dirichlet characters. For a fundamental discriminant D, let χ_D be the quadratic character modulo $|D|$ given by the Kronecker symbol, i.e,

$$\chi_D(n) := \left(\frac{D}{n} \right).$$

In 1951, S. Chowla and P. Erdős [9] proved that the limit as N tends to infinity of the frequencies

$$\frac{\#\left\{ D : |D| \leq N, \, L(1, \chi_D) \leq x \right\}}{N}$$

exists for all real x and is a continuous distribution function. In the 1960s, Barban [3, 4] calculated moments of $L(1, \chi_D)$ for all integer orders $k > 0$ and thus, showed that the characteristic function of the corresponding distribution has the form

$$\sum_{k=0}^{\infty} \frac{r(k)}{k!} \, (it)^k.$$

Here

$$r(k) = \sum_{\substack{n=1, \\ n \text{ odd}}}^{\infty} \frac{\phi(n) \, \tau_k(n^2)}{n^3},$$

where $\tau_k(n)$ is the kth divisor function, i.e., number of ways of writing n as a product of k natural numbers. This was also obtained by P. D. T. A. Elliott [11, Theorem 22.1] exploiting the multiplicative nature of quadratic characters. More specifically, Elliott proved that

Theorem 1.1. *There is a distribution function $F(z)$ so that*

$$\nu_x \left(D; h(-D) \leq \frac{e^z}{\pi} \sqrt{D} \right) = F(z) + O\left(\sqrt{\frac{\log \log x}{\log x}} \right)$$

holds uniformly for all real z and real $x \geq 9$. $F(z)$ has a probability density, may be differentiated any number of times and has the characteristic function

$$\prod_p \left(\left(\frac{1}{p} + \frac{1}{2} \left(1 - \frac{1}{p} \right) \right)^{1-it} + \frac{1}{2} \left(1 - \frac{1}{p} \right) \left(1 + \frac{1}{p} \right)^{-it} \right)$$

which belongs to the Lebesgue class $L(-\infty, \infty)$.

By Dirichlet's class number formula, for $D \geq 4$,

$$L(1, \chi_{-D}) = \frac{\pi}{\sqrt{D}} \, h(-D).$$

Thus, Elliott's theorem proves that the distribution function of the values $L(1, \chi_{-D})$ is smooth. Further work in this context was carried out by A. Granville and K. Soundararajan [12] where in they established the conjectures of Montgomery and Vaughan regarding the distribution of the extreme values of $L(1, \chi_D)$.

Pursuing the analogy of Erdős functions as non-multiplicative analogs of quadratic Dirichlet characters, one may ask if certain properties of quadratic Dirichlet characters are also satisfied by Erdős functions. In this paper, we follow the approach of Barban to understand the distribution of special values of L-series attached to Erdős functions. For a positive integer q, let E_q be the set of Erdős functions mod q. Note that the convergence condition $\sum_{a=1}^{q} f(a) = 0$ implies that E_q is non-empty only when $q \geq 3$ is odd.

Let B_m denote the mth Bernoulli number defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m.$$

For any positive integer n, let P_n denote the partially ordered set of partitions of n, i.e., for partitions $\underline{\lambda}$ and $\underline{\eta}$ of n, $\underline{\eta} \leq \underline{\lambda}$ if the parts of $\underline{\eta}$ can be obtained by merging the parts of $\underline{\lambda}$. The symbol $\binom{\lambda}{\eta}$ counts the number of ways in which parts of $\underline{\lambda}$ can be merged to obtain $\underline{\eta}$. For each $\underline{\lambda} \in P_n$, we inductively define $c(\underline{\lambda})$ as follows.

$$c((n)) := 2^{2nk}((k-1)!)^{2n}\left((-1)^{nk+1}\frac{B_{2nk}}{2nk!}\right)$$

and

$$c(\underline{\lambda}) = ((\lambda_1, \cdots, \lambda_m)) := \left[2^{2nk}((k-1)!)^{2n}\left(\prod_{i=1}^{m}(-1)^{\lambda_i k+1}\frac{B_{2\lambda_i k}}{2\lambda_i k!}\right)\right] - \sum_{\underline{\eta} < \underline{\lambda}}\binom{\lambda}{\eta}c(\underline{\eta}). \qquad (1)$$

Then, using the method of moments, we show that

Theorem 1.2. *Fix a positive integer $k \geq 1$. For any integer $r \geq 1$ and real x, let*

$$E_{2r+1}^{(k)} := \{f \in E_{2r+1} : f \text{ is of the same parity as } k\}$$

and

$$F_r(x) := \frac{\#\{f \in E_{2r+1}^{(k)} : L(k, f) \leq x\}}{\#E_{2r+1}^{(k)}}.$$

Then, $F_r(x)$ converges to a distribution $F(x)$ at every point of continuity of the latter. Moreover, the corresponding characteristic function is entire and is given by

$$\phi(t) = \sum_{n=0}^{\infty} \frac{M(2n)}{(2n)!}(-t)^n,$$

where

$$M(2n) := \frac{\pi^{2nk}}{((k-1)!)^{2n} 2^{2n}}\left(\sum_{\underline{\lambda} \in P_n} c(\underline{\lambda})\right).$$

The restriction on the parity of the functions is inherent in the nature of the value $L(1, f)$ for any periodic function f. Indeed, if f is a q-periodic rational valued function with $\sum_{a=1}^{q} f(a) = f(q) = 0$, then Gauss's formula for the digamma function (see [16] for details) gives

$$L(1, f) = \frac{-\pi}{2q}\sum_{a=1}^{q-1} f(a)\cot\left(\frac{a\pi}{q}\right) + \frac{2}{q}\sum_{0 < j \leq q/2}\log\sin\frac{\pi j}{q}\sum_{a=1}^{q-1} f(a)\cos\left(\frac{2\pi aj}{q}\right).$$

Therefore, when f is odd, the second term in the above expression vanishes and $L(1, f)$ simplifies to an algebraic multiple of π. On the other hand, if f is not odd, then $L(1, f)$ is a linear form in logarithms of algebraic numbers making it intractable, especially in order to use the method of moments. Another difficulty that arises in studying the nature of these

values as opposed to those of Dirichlet L-functions is the absence of multiplicativity among the coefficients. Thus, the probabilistic method of Elliott does not apply to this scenario.

Although some progress has been made towards Conjecture 1, it remains open in the cases $q \equiv 1 \bmod 4$ or $q > 2\phi(q) + 1$. Conjecture 1 follows from a theorem of Baker, Birch and Wirsing [1] when q is prime. It was proved for $q < 2\phi(q) + 1$ by T. Okada [22] and for $q \equiv 3 \bmod 4$ by M. Ram Murty and N. Saradha [20]. In 2015, T. Chatterjee and M. Ram Murty [6] approached Conjecture 1 from a density theoretic perspective. They showed that if

$$S(x) := \#\big\{ q \equiv 1 \bmod 4, \ q \le x \ : \ \text{Erdős's conjecture holds for } q \big\},$$

then

$$\lim_{x \to \infty} \frac{S(x)}{x/4} \ge 0.82.$$

This can be interpreted as the Erdős conjecture being true for at least 82% of $q \equiv 1 \bmod 4$, which is the best possible lower bound using their methods. In this paper, we use an alternate approach and improve on their result by proving that

Theorem 1.3. *Let $q \ge 3$ be an odd positive integer. Let E_q be the set of Erdős functions mod q and $V_q := \big\{ f \in E_q \ : \ L(1, f) = 0 \big\}$. Then,*

$$\lim_{x \to \infty} \left\{ \left(\sum_{\substack{3 \le q \le x, \\ q \ odd}} \#V_q \right) \bigg/ \left(\sum_{\substack{3 \le q \le x, \\ q \ odd}} \#E_q \right) \right\} = 0.$$

This shows that Conjecture 1 is true with "probability" one.

Another aspect of this question is the non-vanishing of special values of $L(s, f)$ for an Erdős function f at positive integers greater than 1. In this direction, we ask the following question.

Question 1.4. *Let $q > 2$ and $k > 1$ be integers and f be an Erdős function mod q. Then is it true that $L(k, f) \ne 0$?*

Remark 1.5. *Let f be an Erdős function mod q and $k > 1$ be an integer. Suppose that $L(k, f) = 0$. Then observe that*

$$|f(1)| = \left| \sum_{n=2}^{\infty} \frac{f(n)}{n^k} \right| \le \zeta(k) - 1.$$

This implies that $2 \le \zeta(k)$, i.e., $k < 2$. This establishes that $L(k, f) \ne 0$ for any Erdős function f if $k \ge 2$. Thus, Erdős's conjecture is interesting when $k = 1$.

2. Preliminaries

In this section, we introduce the results to be used later.

2.1. L-series attached to periodic functions.
Let q be a fixed positive integer and $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. Consider $f : \mathbb{Z} \to \overline{\mathbb{Q}}$, periodic with period q. Define

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Observe that $L(s, f)$ converges absolutely for $\Re(s) > 1$. Since f is periodic,

$$L(s, f) = \sum_{a=1}^{q} f(a) \sum_{k=0}^{\infty} \frac{1}{(a + kq)^s}$$

$$= \frac{1}{q^s} \sum_{a=1}^{q} f(a) \zeta(s, a/q),$$

where $\zeta(s, x)$ is the Hurwitz zeta-function. For $\Re(s) > 1$ and $0 < x \le 1$, the Hurwitz zeta-function is defined as

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}.$$

In 1882, Hurwitz [15] obtained the analytic continuation and functional equation of $\zeta(s, x)$. He proved:

Theorem 2.1. *The Hurwitz zeta-function, $\zeta(s, x)$ extends analytically to the entire complex plane except for a simple pole at $s = 1$ with residue 1. In particular,*

$$\zeta(s, x) = \frac{1}{s - 1} - \Psi(x) + O(s - 1), \tag{2}$$

where Ψ is the Digamma function, which is the logarithmic derivative of the gamma function.

Thus, $\sum_{n=1}^{\infty} f(n)/n$ exists whenever $\sum_{a=1}^{q} f(a) = 0$, which we will assume henceforth. This makes $L(s, f)$ an entire function. Moreover, $L(1, f)$ can be expressed as a combination of values of the Digamma function. Using (2) we get,

$$L(1, f) = -\frac{1}{q} \sum_{a=1}^{q} f(a) \Psi\left(\frac{a}{q}\right). \tag{3}$$

Taking the logarithmic derivative of the identity $\Gamma(x)\Gamma(1 - x) = \pi/\sin(\pi x)$, we obtain that

$$\Psi(x) - \Psi(1 - x) = \pi \cot(\pi x).$$

Therefore, if f is an odd function, we can pair the terms corresponding to a and $q - a$ in (3) to get that

$$L(1, f) = \frac{\pi}{q} \sum_{a=1}^{(q-1)/2} f(a) \cot\left(\frac{a\pi}{q}\right),$$

as $\cot(\pi(1 - x)) = -\cot \pi x$, $f(q - a) = -f(a)$ and $\cot(\pi/2) = 0$. Similarly, $L(k, f)$ can be evaluated if k and f have the same parity, i.e., if both k and f are either odd or even. In particular,

$$L(k, f) = -\frac{(-1)^k}{(k - 1)! \, q^k} \sum_{a=1}^{(q-1)/2} f(a) \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (\pi \cot \pi z) \bigg|_{z=a/q}\right). \tag{4}$$

For a proof of the above fact, we refer the reader to [19, Theorem 10].

2.2. **Higher dimensional Dedekind sums.** In the course of evaluating moments of $L(k, f)$, we also encounter a generalization of the higher dimensional Dedekind sums which were introduced by D. Zagier [27]. These sums, studied by A. Bayad and A. Raouj [2], are defined

as follows. For $i = 0, \cdots, d$, let a_0 be a positive integer, $a_1, \cdots, a_d$ be positive integers co-prime to a_0 and $m_0, \cdots, m_d$ be non-negative integers. Define

$$
C(a_i \, ; \, a_0, \cdots, \widehat{a_i}, \cdots, a_d \, | \, m_i \, ; \, m_0, \cdots, \widehat{m_i}, \cdots, m_d)
$$

$$
:= \begin{cases} \dfrac{1}{a_i^{m_i+1}} \sum_{k=1}^{a_i - 1} \prod_{\substack{j=0, \\ j \neq i}}^{d} \left(\dfrac{d^{m_j}}{dz^{m_j}} (\cot z) \right) \Bigg|_{z = \pi k a_j / a_i} , & \text{if } a_i \geq 2, \\[4mm] 0 & \text{if } a_i = 1. \end{cases} \tag{5}
$$

Here $\widehat{x_n}$ means that the term x_n is omitted. It can be shown that the number given by (5) is in fact rational. For a proof of this fact and further properties of the higher dimensional Dedekind sums, see [27] and [2]. Moreover, these sums satisfy a reciprocity law given by:

Theorem 2.2. [2, Theorem 2.0.2] *Let d be a positive integer, $a_0, \cdots, a_d$ be pairwise positive integers and $m_0, \cdots, m_d$ be non-negative integers. Let B_n denote the nth Bernoulli number. Assume that the integer $M = d + m_0 + \cdots + m_d$ is even. Then we have*

$$
\sum_{i=0}^{d} (-1)^{m_i} \, m_i! \sum_{\substack{l_0, \cdots, \widehat{l_i}, \cdots, l_d \geq 0 \\ l_0 + \cdots + \widehat{l_i} + \cdots + l_d = m_i}} \left(\prod_{\substack{j=0 \\ j \neq i}}^{d} \frac{a_j^{l_j}}{l_j!} \right)
$$

$$
\times \, C(a_i \, ; \, a_0, \cdots, \widehat{a_i}, \cdots, a_d \, | \, m_i \, ; \, m_0 + l_0, \cdots, \widehat{m_i + l_i}, \cdots, m_d + l_d)
$$

$$
= \begin{cases} -\left(R + (-1)^{d/2} \right) & \text{if all } m_i \text{ are zero}, \\ -R & \text{otherwise}, \end{cases}
$$

where [1]

$$
R = \frac{(-1)^{M/2} \, 2^M}{\prod_{i=0}^{d} a_i^{m_i+1}} \sum_{\substack{j_0, \cdots, j_d \geq 0 \\ j_0 + \cdots + j_d = M/2}} \prod_{i=0}^{d} a_i^{2j_i} A_{i, j_i}
$$

and

$$
A_{i, j_i} = \begin{cases} \dfrac{B_{2j_i}}{(2j_i - 1 - m_i)!(2j_i)} & \text{if } j_i \text{ is an integer } \geq (m_i + 1)/2, \\[3mm] (-1)^{m_i} \, m_i! & \text{if } j_i = 0, \\[1mm] 0 & \text{otherwise}. \end{cases}
$$

2.3. **Partial sum lemma.** We will use the following elementary exercise from [23, Problem 70, pg. 16].

Lemma 2.3. *Let the sequences a_n and b_n satisfy the conditions:*

$$
b_n > 0, \quad n = 1, 2, \cdots \, ; \quad b_1 + b_2 + b_3 + \cdots + b_n + \cdots \text{ diverges};
$$

and

$$
\lim_{n \to \infty} \frac{a_n}{b_n} = s.
$$

Then

$$
\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = s.
$$

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[1]The minus sign in front of the right hand side is missing in the statement of [2, Theorem 2.0.2] but is evident from the proof.

3. Proof of Theorems

3.1. **Proof of Theorem 1.2.** Our proof is based on the following lemma from [5, Theorem 30.2, pg. 390].

Lemma 3.1. *Suppose the distribution of X is determined by its moments, that the X_n have moments of all orders and that $\lim_n E[X_n^r] = E[X^r]$ for $r = 1, 2, \cdots$. Then $X_n \Rightarrow X$, i.e., the distribution of X_n converges to the distribution of X wherever the distribution of X is continuous.*

Thus, we first show that

$$M(n) := \lim_{r \to \infty} m_{2r+1}(n) = \lim_{r \to \infty} \frac{\sum_{f \in E_{2r+1}^{(k)}} L(k, f)^n}{\# E_{2r+1}^{(k)}}$$

is finite. In order to show that the limiting distribution is determined by its moments, we use a general theorem from [5, Theorem 30.1, pg. 388] stated below.

Theorem 3.2. *Let μ be a probability measure on the line having finite moments $\alpha_k = \int_{-\infty}^{\infty} x^k \mu(dx)$ of all orders. If the power series $\sum_{k=0}^{\infty} \alpha_k r^k/k!$ has a positive radius of convergence, then μ is determined by its moments.*

To begin with, we evaluate

$$m_q(n) := \frac{\sum_{f \in E_q^{(k)}} L(k, f)^n}{\# E_q^{(k)}},$$

for any non-negative integer n in terms of sums of the form (5). By (4),

$$\sum_{f \in E_q^{(k)}} L(k, f)^n$$

$$= \frac{(-1)^{kn}}{\left((k-1)!\right)^n q^{kn}} \sum_{a_1, \cdots, a_n = 1}^{(q-1)/2} \prod_{j=1}^{n} \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (\pi \cot \pi z) \Big|_{z = a_j/q} \right) \sum_{f \in E_q^{(k)}} f(a_1) \cdots f(a_n),$$

$$= \frac{(-1)^{kn} \pi^k}{\left((k-1)!\right)^n q^{kn}} \sum_{a_1, \cdots, a_n = 1}^{(q-1)/2} \prod_{j=1}^{n} \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (\cot z) \Big|_{z = a_j/q} \right) \sum_{f \in E_q^{(k)}} f(a_1) \cdots f(a_n).$$

Note that if $f \in E_q^{(k)}$ then $-f \in E_q^{(k)}$. Thus, if n is odd, the inner sum becomes zero by pairing terms corresponding to f and $-f$. Therefore, $m_q(n) = 0$ when n is odd.

Henceforth, let n be even and $p(n)$ denote the number of partitions of n. The above sum can be partitioned into $p(n)$ many sums according to the equality of the indices $a_1, \cdots, a_n$. In particular, for a partition $\underline{\lambda} = (\lambda_1, \cdots, \lambda_m)$ of n, the inner sum becomes

$$\sum_{f \in E_q^{(k)}} f(a_1)^{\lambda_1} \cdots f(a_m)^{\lambda_m}, \tag{6}$$

where $1 \leq a_1, \cdots, a_m \leq q - 1$ are all distinct. Clearly, the above sum is $\# E_q^{(k)}$ when λ_l is even for all $1 \leq l \leq m$. Now, without loss of generality, suppose that λ_1 is odd. Since even and odd functions are determined by their values on $1 \leq a \leq r$, for any $f \in E_q^{(k)}$, there is a unique $f^- \in E_q^{(k)}$ such that

$$f^-(n) = \begin{cases} f(n) & \text{if } n \neq a_1, \\ -f(n) & \text{if } n = a_1, \end{cases}$$

for $1 \leq n \leq r$. Pairing up the terms corresponding to f and f^- in (6), we obtain that

$$\sum_{f \in E_q^{(k)}} f(a_1)^{\lambda_1} \cdots f(a_m)^{\lambda_m} = \begin{cases} \#E_q^{(k)}, & \text{if } \lambda_1, \cdots, \lambda_m \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, only those terms corresponding to partitions consisting of even parts survive and one can write

$$\sum_{f \in E_q^{(k)}} L(k, f)^n$$

$$= \frac{\pi^k}{\left((k-1)!\right)^{2n} 2^{2n} q^{2kn}} \sum_{\substack{\underline{\lambda} = (\lambda_1, \cdots, \lambda_m), \\ \underline{\lambda} \in P_n}} \sum_{a_1, \cdots, a_m = 1}^{q-1} {}' \prod_{j=1}^{m} \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (\cot z) \Big|_{z=a_j/q} \right)^{2\lambda_j},$$

where $\sum'$ denotes that the sum is taken over distinct $1 \leq a_1, \cdots, a_m \leq q-1$. The inner sum can be expressed in terms of generalized higher dimensional Dedekind sums as follows. For any positive integer u, define

$$S_{q,k}^{(u)} := q\, C(q\,;\, \underbrace{1, 1, \cdots, 1}_{2u \text{ times}} \,|\, 0\,;\, \underbrace{k-1, \cdots, k-1}_{2u \text{ times}}).$$

With the notation as in Section 1, define $\mathfrak{S}_{q,k}^{(\eta)}$ inductively as follows. $\mathfrak{S}_{q,k}^{(n)} := S_{q,k}^{(n)}$ and for any $\underline{\lambda} = (\lambda_1, \cdots, \lambda_m)$,

$$\mathfrak{S}_{q,k}^{(\underline{\lambda})} := S_{q,k}^{(\lambda_1)} \cdots S_{q,k}^{(\lambda_m)} - \sum_{\underline{\eta} \leq \underline{\lambda}} \binom{\underline{\lambda}}{\underline{\eta}} \mathfrak{S}_{q,k}^{(\underline{\eta})}. \tag{7}$$

Thus, we have that

$$m_q(2n) = \frac{\pi^k}{\left((k-1)!\right)^{2n} 2^{2n} q^{2kn}} \left\{ \sum_{\underline{\lambda} \in P_n} \mathfrak{S}_{q,k}^{\underline{\lambda}} \right\}.$$

To understand the asymptotic behaviour of $m_q(2n)$ as $q \to \infty$, we use the explicit evaluation of $S_{q,k}^{(u)}$ using Theorem 2.2. Thus,

$$S_{q,k}^{(u)} = \sum_{t=1}^{q-1} \left(\frac{d^{k-1}}{dz^{k-1}} (\cot z) \right) \Big|_{z=\pi t/q}^{2u}$$

$$= \begin{cases} -q\left(R + (-1)^u\right) & \text{if } k = 1, \\ -q\,R & \text{otherwise,} \end{cases}$$

where

$$R = \frac{(-1)^{uk}\, 2^{2uk}}{q} \sum_{\substack{j_0, \cdots, j_{2u} \\ j_0 + \cdots + j_{2u} = uk}} \alpha_{j_0, \cdots, j_{2u}}\, q^{2j_0},$$

for $\alpha_{j_0, \cdots, j_{2u}} \in \mathbb{Q}$ given by Theorem 2.2. In particular, $S_{q,k}^{(u)}$ is a polynomial in q of degree $2uk$ with leading coefficient $-(-4)^{uk} \alpha_{uk,0,\cdots,0}$, given explicitly by Theorem 2.2. Thus,

$$S_{q,k}^{(u)} \sim \left(2^{2uk} \left((k-1)!\right)^{2u} (-1)^{uk+1} \frac{B_{2uk}}{(2uk)!} \right) q^{2uk}, \quad \text{as } q \to \infty.$$

Using this, (7) and the definition of $c(\underline{\lambda})$ (1), we get that

$$\mathfrak{S}_{q,k}^{(\underline{\lambda})} \sim c(\underline{\lambda})\, q^{2nk},$$

as q tends to infinity. Hence,

$$M(2n) = \lim_{q\to\infty} m_q(2n) = \frac{\pi^{2nk}}{((k-1)!)^{2n}\, 2^{2n}} \left(\sum_{\underline{\lambda}\in P_n} c(\underline{\lambda}) \right).$$

Since the limit as q tends to infinity of $m_q(2n)$ exists, by Lemma 3.1, there exists a limiting distribution $F(x)$ whose odd moments are zero and even moments are given by $M(2n)$. Thus, the characteristic function of $F(x)$ is given by

$$\phi(t) := \sum_{n=0}^{\infty} \frac{M(2n)}{(2n)!}\, (it)^{2n}.$$

By Lemma 3.1 and Lemma 3.2, it suffices to show that $\phi(t)$ has positive radius of convergence. In fact, we prove that it is entire.

Let $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ for $\mathfrak{R}(s) > 1$ be the Riemann zeta function. In 1737, Euler showed that

$$\zeta(2m) = (-1)^{m+1} \frac{B_{2m}\, (2\pi)^{2m}}{(2m)!\, 2}.$$

Since $\zeta(2m+2) < \zeta(2m)$,

$$\left| \frac{B_{2m+2}}{(2m+2)!} \right| < \left| \frac{B_{2m+2}\, \pi^2}{(2m+2)!} \right| < \left| \frac{B_{2m}}{(2m)!} \right|.$$

Hence,

$$c(\underline{\lambda}) \le \left| 2^{2nk}((k-1)!)^{2n} \left(\prod_{i=1}^{m} (-1)^{\lambda_i k+1} \frac{B_{2\lambda_i k}}{(2\lambda_i k)!} \right) \right| \le 2^{2nk}((k-1)!)^{2n} \left| \frac{B_{2k}}{(2k)!} \right|^m.$$

and thus,

$$M(2n) = \frac{\pi^{2nk}}{((k-1)!)^{2n}\, 2^{2n}} \left(\sum_{\underline{\lambda}\in P_n} c(\underline{\lambda}) \right) \le \frac{(2\pi)^{2nk}}{2^{2n}}\, p(n) \left| \frac{B_{2k}}{(2k)!} \right|^n = \frac{p(n)}{2^n} \left(\frac{\zeta(2k)}{2} \right)^n,$$

where $p(n)$ denotes the number of partitions of n. In 1918, Hardy and Ramanujan [13] showed that

$$p(n) \sim \frac{1}{(4\sqrt{3})\, n} \exp\left(\pi \sqrt{\frac{2n}{3}} \right) \text{ as } n \to \infty.$$

Now using Stirling's formula and the asymptotics for $p(n)$, we obtain that

$$\left(\frac{M(2n)}{(2n)!} \right)^{1/2n} \ll e^{-c\log n},$$

for a positive constant c. Therefore, applying the root test gives that the radius of convergence of $\phi(t)$ is infinite. This proves the theorem.

3.2. **Proof of Theorem 1.3.** Let $q \geq 3$ be odd and E_q be the set of all Erdős functions mod q. Let $r := (q-1)/2$. We define a relation on this set as follows. For $f, g \in E_q$,

$$f \sim g \iff f(a) = g(a), \ \forall 1 \leq a \leq q, \ (a, q) \neq 1.$$

One can easily check that $\sim$ is an equivalence relation. Before proceeding, we prove the following proposition.

Proposition 3.3. *There exists at most one Erdős function f in every equivalence class of E_q under $\sim$, such that $L(1, f) = 0$.*

Proof. Suppose $f, g \in E_q$ are such that $L(1, f) = L(1, g) = 0$. Thus, by (3) and the convergence condition, we have

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} f(a) \left[\Psi\left(\frac{a}{q}\right) + \gamma \right] = - \sum_{\substack{a=1 \\ (a,q) \neq 1}}^{q} f(a) \left[\Psi\left(\frac{a}{q}\right) + \gamma \right]$$

$$= - \sum_{\substack{a=1 \\ (a,q) \neq 1}}^{q} g(a) \left[\Psi\left(\frac{a}{q}\right) + \gamma \right]$$

$$= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} g(a) \left[\Psi\left(\frac{a}{q}\right) + \gamma \right].$$

Therefore, we obtain

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left[f(a) - g(a) \right] \left[\Psi\left(\frac{a}{q}\right) + \gamma \right],$$

which is a $\mathbb{Q}$-linear relation among the numbers

$$\Psi(a/q) + \gamma, \ \ 1 \leq a \leq q, \ (a, q) = 1.$$

But these numbers are $\mathbb{Q}$-linearly independent as proven in [20, Theorem 4]. Hence, $f = g$. $\square$

Therefore, it suffices to count the number of equivalence classes of E_q under $\sim$. In order to count these, note that each equivalence class differs from the other based on the values of functions on $N_q := \{ a : 1 \leq a < q, (a, q) \neq 1 \}$. At each point in this set, an Erdős function can take either 1 or -1, with the only restriction that

$$\#\{a \in N_q : f(a) = 1\} = \#\{a \in N_q : f(a) = -1\} = \frac{q - 1 - \phi(q)}{2}.$$

Owing to this, two cases arise. For simplicity of notation, let $n_q := (q - 1 - \phi(q))/2$ and recall that $r = (q-1)/2$.

(a) $r \geq n_q$: In this case, the number of $a \in N_q$ where a function takes the value 1 ranges from 0 to n_q. Thus, the total number of equivalence classes is

$$\left| E_q / \sim \right| = \sum_{k=0}^{n_q} \binom{n_q}{k} = 2^{n_q}.$$

(b) $r < n_q$: Let $j := n_q - r$. Then, the number of $a \in N_q$ where a function takes the value 1 has to be at least j. Hence, the number of equivalence classes is

$$\left| E_q / \sim \right| = \sum_{k=j}^{n_q} \binom{n_q}{k} < 2^{n_q}.$$

Therefore, in either case $|V_q| \le 2^{(q-1-\phi(q))}$. Now, note that

$$\left|E_q\right| = \binom{2r}{r}.$$

Using the bounds by [25], one has

$$\sqrt{2\pi}\, n^{(n+\frac{1}{2})}\, e^{-n} < n! < \sqrt{2\pi}\, e\, n^{(n+\frac{1}{2})}\, e^{-n},$$

for all $n \in \mathbb{N}$, we get that

$$\binom{2r}{r} = \frac{(2r)\,!}{(r!)^2} \ge \frac{\sqrt{2\pi}\,(2r)^{(2r+1/2)}\,e^{-2r}}{\left(\sqrt{2\pi}\,e\,r^{(r+1/2)}\,e^{-r}\right)^2} = \frac{\sqrt{2}\,2^{2r}\,r^{2r}\,\sqrt{r}}{\sqrt{2\pi}\,e^2\,r^{2r}\,r} = \frac{2^{2r}}{e^2\sqrt{\pi}\,\sqrt{r}}.$$

Thus,

$$\left(\sum_{\substack{3\le q\le x, \\ q\ \text{odd}}} \#V_q\right)\Bigg/\left(\sum_{\substack{3\le q\le x, \\ q\ \text{odd}}} \#E_q\right) \ll \frac{\sum_{r\le x} 2^{2r-\phi(2r+1)}}{\sum_{r\le x}(2^{2r}/\sqrt{r})} \ll \frac{\sum_{r\le x} 2^{2r-(2r/(\log\log r))}}{\sum_{r\le x}(2^{2r}/\sqrt{r})}$$

because by [14, pg. 217],

$$\liminf_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-C}.$$

Therefore, by Lemma 2.3, the right hand side tends to zero as $x \to \infty$. This proves Theorem 1.3.

4. Conclusion

Theorem 1.2 leads to various natural questions regarding the distribution function $F(x)$. For example, the continuity of F, the rate of convergence in distribution as well as the tail of the distribution are but a few problems arising out of our study. Another direction is to understand the distribution of $L(k, f)$ without any restriction on the parity of Erdős functions. In this course, one is led to study the analogue of higher dimensional Dedekind sums for the digamma function. Since these investigations are far afield from our current focus, we relegate them to future research.

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