ON VALUES OF THE DIGAMMA FUNCTION AT RATIONAL ARGUMENTS

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Abstract. For a rational number a/N with $1 \leq a < N$, we call the value of the digamma function, $\psi(a/N)$, an N-division value. An N-division value is said to be primitive if in addition, $(a, N) = 1$. In this paper, we derive an explicit expression for an N-division value as a rational linear combination of primitive N-division values and logarithms of primes that divide N. For a periodic arithmetic function f, let $L(s, f) = \sum_{n \geq 1} f(n)\, n^{-s}$. Due to the connection between division values of the digamma function and $L(1, f)$, our approach leads to new observations regarding the latter.

1. Introduction

Since the time of Dirichlet, we have known that special values of L-series encode deep arithmetic information. Thus, the study of such special values occupies a central place in number theory. In the case of Dirichlet series arising from periodic arithmetical functions (such as Dirichlet characters), these special values turn out to be linear combinations of polygamma functions evaluated at rational numbers. Hence, the interest is in studying relations among special values of the digamma function at rational arguments.

The digamma function $\psi(z)$ is the logarithmic derivative of the Γ-function, and is, as alluded to above, an important special function appearing in number theory. On logarithmic differentiation of the Hadamard product expansion of the Γ-function, we get

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right), \tag{1}$$

where γ denotes the Euler's constant. Thus, $\psi(z)$ has poles at non-positive integers $z = 0, -1, -2, ...$ and when $z = 1$, the series on the right hand side of (1) telescopes and we deduce that $\psi(1) = -\gamma$. More generally, if m is a natural number, then the right hand side of (1) again telescopes and we get that

$$\psi(m) = -\gamma + \sum_{j=1}^{m-1} \frac{1}{j}.$$

Date: June 25, 2026.

2020 *Mathematics Subject Classification.* 11J72, 11R18.

Key words and phrases. Digamma function, cotangent function.

Research of the first author is supported by Coleman Postdoctoral Fellowship. Research of the second author is partially supported by an NSERC Discovery grant. Research of the third author is partially supported by the INSPIRE Faculty Fellowship.

The digamma function directly inherits the following relations from the Γ-function:

$$\begin{aligned} \psi(z+1) &= \psi(z) + \frac{1}{z}, \\ \psi(1-z) &= \psi(z) + \pi \cot \pi z, \\ \psi(mx) &= \frac{1}{m} \sum_{k=0}^{m-1} \psi\left(x + \frac{k}{m}\right) + \log m, \text{ for any natural number } m. \end{aligned} \tag{2}$$

In this paper, we study linear relations among values of the digamma function at rational arguments. To this end, we say that any value $\psi(a/N)$ with $1 \leq a < N$ is an *N-division value* of the digamma function. We say that it is a *primitive N-division value* if in addition $(a, N) = 1$. One of our initial goals is to explore if any N-division value can be expressed in terms of primitive N-division values. We utilize two approaches in this study; one via certain constants defined by Okada [12]; and the other via an induction operator introduced by the first and third author in [3].

Towards this goal, let $\mathcal{M}_N$ be the monoid generated by the prime factors of N. For $1 \leq r, n \leq N$ with $(n, N) = 1$, define the *Okada constants* to be the numbers

$$A(r, n) = \sum_{\substack{b \in \mathcal{M}_N, \\ bn \equiv r \bmod N}} \frac{1}{b}. \tag{3}$$

Note that for $\mathrm{Re}(s) > 0$, we have the series

$$\sum_{b \in \mathcal{M}_N} \frac{1}{b^s} = \prod_{p \mid N} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{4}$$

Thus, the series appearing in the definition of the numbers $A(r, n)$ is absolutely convergent. These constants first appeared in the work of T. Okada (see [11], [12]) and have been isolated for their arithmetic significance by the second and the third author in [10]. Among other properties, it is known that $A(r, n)$ are *rational numbers*.

We first establish the following result deriving an expression for any N-division value in terms of primitive N-division values and logarithms of primes dividing N.

Theorem 1.1. *Let $1 \leq \delta < N$ with $(\delta, N) = d$ and for $1 \leq n < N$ with $(n, N) = 1$, let $A(\delta, n)$ be as above. Then*

$$\psi\left(\frac{\delta}{N}\right) = \sum_{\substack{n=1, \\ (n,N)=1}}^{N} A(\delta, n)\, \psi\left(\frac{n}{N}\right) + \log d + \sum_{\substack{p \mid N, \\ p \nmid (N/d) \\ p \text{ prime}}} \frac{\log p}{p-1}.$$

This theorem suggests that we study the relationship between the N-division values and the q-division values of the digamma function, when q is a divisor of N. In this context, define

$$\mathscr{D}_P(N) := \mathbb{Q}\left\langle \left\{ \psi\left(\frac{a}{N}\right) \ : \ (a, N) = 1, \, 1 \leq a < N \right\} \right\rangle,$$

that is, $\mathscr{D}_P(N)$ is the $\mathbb{Q}$-vector space spanned by the primitive N-division values of the digamma function. In [9], the second author and Saradha conjectured that $\dim_{\mathbb{Q}} \mathscr{D}_P(N) = \varphi(N)$. This is still open.

Our approach to Theorem 1.1 allows us to establish the following result towards this conjecture.

Corollary 1.1. *There exists at most one natural number N for which $\dim_{\mathbb{Q}} \mathscr{D}_P(N) < \varphi(N)$. In other words, $\dim_{\mathbb{Q}} \mathscr{D}_P(N) = \varphi(N)$ with at most one exception.*

The above statement is the main result in [6], proved by Chatterjee and Dhillon using different methods.

Since the digamma function is related to the cotangent function via the functional equation (2), by virtue of Theorem 1.1, we deduce the following.

Corollary 1.2. *If $1 \le \delta < N/2$, then*

$$\cot \frac{\delta\pi}{N} = - \sum_{\substack{n=1 \\ (n,N)=1}}^{N} A(\delta, n) \cot\left(\frac{n\pi}{N}\right).$$

In [7], Girstmair used the theory of group rings to study relations between special values of the cotangent function. To state his result, we require the following notation. For a positive integer n and a prime p, let $\nu_p(n)$ denote the largest non-negative integer k such that $p^k \mid n$. Let

$$n^* = \prod_{p \mid n} p \qquad \text{and} \qquad \widehat{n} = \prod_{p \mid n} \left(p^{\nu_p(n)} - 1\right).$$

For any divisor d of n and prime p not dividing d, let $o(p, d)$ be the order of $p \bmod d$ and

$$\lambda_{n,d} = \prod_{\substack{p \mid n \\ p \nmid d}} p^{o(p,d)}.$$

Girstmair's result ([7, Theorem 1] with $r = 1$) can now be stated as follows.

Theorem (K. Girstmair)**.** *Let $N \ge 3$ be a natural number and $q \ge 2$ be a divisor of N. Then*

$$\cot\left(\frac{\pi}{q}\right) = \sum_{\substack{1 \le n < N/2, \\ (n,N)=1}} a_n(q) \cot\left(\frac{n\pi}{N}\right),$$

where

$$a_n(q) = \left(\left(\frac{N}{q}\right) \widehat{\lambda_{N,q}}\right)^{-1} \left(\sum_{\substack{d \equiv n \bmod q, \\ \lambda^*_{N,q} \mid d \mid \lambda_{N,q}}} d \; - \sum_{\substack{d \equiv -n \bmod q, \\ \lambda^*_{N,q} \mid d \mid \lambda_{N,q}}} d \right). \tag{5}$$

Girstmair also derived a similar expression for $\cot(\delta\pi/q)$ when $1 \le \delta < q$ with $(\delta, q) = 1$ in terms of $\cot(n\pi/N)$, $1 \le n < N/2$ with $(n, N) = 1$.

In 1949, Siegel (see Chowla [5]) proved that the numbers

$$\cot \frac{n\pi}{N}, \qquad 1 \le n \le N/2, \quad (n, N) = 1$$

are linearly independent over $\mathbb{Q}$. Thus, when $\delta \mid N$, we can compare the expressions obtained in Corollary 1.2 with (5) and write $A(\delta, n) + A(\delta, N - n)$ as finite sums involving divisors of N. In particular, we have the following.

Corollary 1.3. *Let $1 < \delta < N/2$ such that $\delta \mid N$. Then for all $1 \le n < N/2$ with $(n, N) = 1$, we get*

$$A(\delta, n) + A(\delta, N - n) = -a_n(N/\delta),$$

with $a_n(N/\delta)$ given by equation (5).

Given the interest in the Okada constants highlighted in [10], we provide an alternate form for them via a generating function. Henceforth, $\zeta_t = e^{2\pi i/t}$ for any natural number $t \geq 2$.

Theorem 1.2. *Let r, n and N be natural numbers, with $q = (r, N)$ and $(n, N) = 1$. Let $\mathcal{G}_{N/q}$ be the Galois group of $\mathbb{Q}(\zeta_{N/q})/\mathbb{Q}$ and $\sigma_c \in \mathcal{G}_{N/q}$ with $\sigma_c(\zeta_{N/q}) = \zeta_{N/q}^c$ for $c \in (\mathbb{Z}/(N/q)\mathbb{Z})^*$. Then we have*

$$\sum_{\substack{b=1 \\ (b,N/q)=1}}^{N/q} A(qb, n)\, \sigma_b \;=\; \frac{\sigma_n}{q} \prod_{\substack{p|N \\ p\nmid(N/q) \\ p \text{ - prime}}} \left(1 - \frac{\sigma_p}{p}\right)^{-1}, \tag{6}$$

as elements of the group ring $\mathbb{Q}[\mathcal{G}_{N/q}]$.

Remark. *The above theorem can be used to compute the values of $A(r, n)$. Let $q = (r, N)$ and for a prime $p \nmid N/q$, let $o(p, N/q)$ be the order of $p \bmod N/q$. For such a prime p, we have*

$$\left(1 - \frac{\sigma_p}{p}\right)^{-1} = \frac{p^{o(p,N/q)}}{p^{o(p,N/q)} - 1} \sum_{i=0}^{o(p,N/q)-1} \frac{\sigma_p^i}{p^i}.$$

Expanding the product on the right hand side of (6) and comparing term by term with the left hand side of (6) determines the constants $A(r, n)$.

The above results are based on a fundamental connection between the digamma values and the value of the Dirichlet series $L(s, f)$ attached to a periodic arithmetic function f at $s = 1$. Recall that given a function f on integers, periodic with period N, the associated Dirichlet series is

$$L(s, f) = \sum_{n\geq 1} \frac{f(n)}{n^s}, \qquad \operatorname{Re} s > 1.$$

It extends to an entire function, provided that

$$\sum_{a=1}^{N} f(a) = 0. \tag{7}$$

In this case, we have (see Section 2):

$$L(1, f) = -\frac{1}{N} \sum_{a=1}^{N} f(a)\, \psi\left(\frac{a}{N}\right). \tag{8}$$

In other words, linear relations among division values of the digamma function are reliant on the vanishing of a certain L-series at $s = 1$. With this connection in mind, we focus on studying the value $L(1, f)$ for functions f modulo N and q, for a divisor q of N.

For a natural number $N \geq 2$, the following $\mathbb{Q}-$vector spaces are relevant to our discussion.

$$\mathbf{F}(N) := \{f : \mathbb{Z} \to \mathbb{Q} \,:\, f(n + N) = f(n) \text{ for all } n \in \mathbb{Z}\}$$

$$\mathbf{F}^{(\mathbf{0})}(N) := \left\{ f \in \mathbf{F}(N) \,:\, \sum_{n=1}^{N} f(n) = 0 \right\}$$

$$\mathbf{F_D}(N) := \{f \in \mathbf{F}(N) \,:\, f(n) = 0 \text{ if } (n, N) > 1\}$$

$$\mathbf{F_D^{(0)}}(N) := \left\{ f \in \mathbf{F_D}(N) \,:\, \sum_{n=1}^{N} f(n) = 0 \right\}$$

The functions in $\mathbf{F_D}(N)$ are said to be of *Dirichlet type* modulo N. Evidently, these can be expressed as a linear combination of Dirichlet characters modulo N. Indeed, if

$$\langle f, \chi \rangle := \frac{1}{\varphi(N)} \sum_{\substack{n=1 \\ (n,N)=1}}^{N} f(n)\,\overline{\chi(n)},$$

then it can be checked that

$$f = \sum_{\chi \bmod N} \langle f, \chi \rangle\, \chi,$$

where the sum is over all Dirichlet characters modulo N.

When $f \in \mathbf{F}^{(\mathbf{0})}(q)$, Theorem 1.1 implies that $L(1, f)$ can be written as a rational linear combination of primitive N-division values and $\log p$ with primes p dividing N, for any N which is a multiple of q. This suggests to define an "induction" operator on the function spaces. More specifically, we have the following.

Definition 1.1. *Let q and N be positive integers greater than 1 such that $q \mid N$. Let $f \in \mathbf{F_D}(q)$ with $f = \sum_{\chi \bmod q} \langle f, \chi \rangle\, \chi$. Define*

$$\mathrm{Ind}_q^N(f) := \sum_{\chi \bmod q} \left(\frac{\langle f, \chi \rangle}{\prod_{p \mid N} (1 - \chi(p) p^{-1})} \right) \chi_N, \tag{9}$$

where χ_N denotes the Dirichlet character mod N induced from $\chi \bmod q$, that is,

$$\chi_N(n) = \begin{cases} \chi(n) & \text{if } (n, N) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

It can be shown that $\mathrm{Ind}_q^N(f) \in \mathbf{F_D}(N)$. When restricted to $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(q)$, the induction operator preserves the value $L(1, f)$. In other words,

$$\mathrm{Ind}_q^N : \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(q) \to \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N) \qquad \text{such that} \qquad L(1, f) = L\left(1, \mathrm{Ind}_q^N(f)\right).$$

See Section 2.4 for details.

A natural question in this context is to identify the image of $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(q)$ in $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$ under Ind_q^N. This, in turn, would allow us to recognize if given a function f periodic modulo N, is $L(1, f) = L(1, g)$ for a function g with period strictly smaller than N.

To address this, define the action of $G = G_N = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ on $\mathbf{F_D}(N)$ as follows. For a function $f \in \mathbf{F_D}(N)$, set $\sigma_n^{-1}(f)(m) = f(mn)$ for all numbers m, where the automorphism $\sigma_n \in G$ is defined by $\sigma_n(\zeta_N) = \zeta_N^n$. Let H be a subgroup of G. Then, we have the subspace of $\mathbf{F_D}(N)$ consisting of functions invariant under the action of H, namely,

$$\mathbf{F_D}(N)^H := \left\{ f \in \mathbf{F_D}(N) : \sigma(f) = f \text{ for all } \sigma \in H \right\}.$$

Correspondingly, $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H = \left\{ f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N) : \sigma(f) = f \text{ for all } \sigma \in H \right\}$.

Another cognate subspace of $\mathbf{F_D}(N)$ (resp. $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$) is

$$\begin{aligned} &\mathbf{F_D}(N)_H \ (\text{ resp. } \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)_H \ ) \\ &:= \left\{ f \in \mathbf{F_D}(N) \ (\text{ resp. } \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)\ ) \ : \ \langle f, \chi \rangle = 0 \text{ for all } \chi \bmod N \text{ such that } \chi|_H \neq 1 \right\}. \end{aligned}$$

Here H is identified with the corresponding subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$ so that $\chi|_H \neq 1$ translates to $\chi(n) \neq 1$ for some n co-prime to N, such that $\sigma_n \in H$, with $\sigma_n(\zeta_N) = \zeta_N^n$. It is not surprising that the above space is related to $\mathbf{F_D}(N)^H$. We will further show that both these spaces can be identified via the value $L(1, f)$.

In order to do so, we require an alternate expression for $L(1, f)$. For $f \in \mathbf{F}(N)$, let

$$\widehat{f}(b) := \frac{1}{N} \sum_{n=1}^{N} f(n)\, \zeta_N^{-bn}$$

be the Fourier transform of f. Then it can be shown that [9, Theorem 19] for $f \in \mathbf{F}^{(\mathbf{0})}(N)$,

$$L(1, f) = -\sum_{b=1}^{N-1} \widehat{f}(b) \, \log\left(1 - \zeta_N^b\right).$$

Furthermore, if $f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$,

$$L(1, f) \in \mathbb{Q}(\zeta_N) \left\langle \left\{ \log \alpha \ : \ \alpha \in \mathcal{O}^*_{\mathbb{Q}(\zeta_N)} \right\} \right\rangle,$$

where $\mathcal{O}^*_{\mathbb{Q}(\zeta_N)}$ denotes the group of units of $\mathbb{Z}[\zeta_N]$. For a proof of this, we refer the reader to Proposition 2.1 in the next section and to [3, Proof of Theorem 3.7].

With the help of this interpretation of the value $L(1, f)$, we deduce the following.

Theorem 1.3. *Let K be a number field contained in $\mathbb{Q}(\zeta_N)$, $\mathcal{O}_K^*$ be the group of units of the ring of integers of K and $H = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/K)$. Then we have*

$$\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H = \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)_H = \left\{ f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N) \ : \ L(1, f) \in K \left\langle \{ \log \alpha \mid \alpha \in \mathcal{O}_K^* \} \right\rangle \right\}.$$

When specialized, the above theorem also gives a necessary and sufficient condition for a function to be in the image of Ind_q^N.

Theorem 1.4. *Let q, $N \geq 2$ be natural numbers such that q divides N. Let $H = \mathrm{Gal}\left(\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_q)\right)$. Then we have*

$$\mathrm{Ind}_q^N \left(\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(q) \right) = \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H = \left\{ f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N) \ : \ L(1, f) \in \mathbb{Q}(\zeta_q) \left\langle \{ \log(\alpha), \alpha \in \mathbb{Z}[\zeta_q]^* \} \right\rangle \right\}.$$

The aim of this paper is to utilize the connection between vanishing of the special value $L(1, f)$ and linear relations among the division values of the digamma function. In particular, Theorem 1.1 can be viewed as an identity which allows one to view 'value at a lower level' in terms of 'primitive values at a higher level'. This is in the same spirit as in [7]. The computation of Okada constants $A(\delta, n)$ which appear in Theorem 1.1 is not straightforward in general. However, Theorem 1.2 provides one approach in this direction. We carry over the theme from Theorem 1.1 to the study of $L(1, f)$ via the induction operator. Theorem 1.3 and 1.4 are then tools used to identify the 'values induced from lower level'. These results can be applied to enhance our understanding of $L(1, f)$, about which several questions still remain unsolved.

Organisation of the paper. In Section 2, we list the prerequisites and obtain an explicit expression for $A(r, n)$ with positive integers r and n, and certain related constants that arise in the course of our proof. We also study relevant properties of the induction operator here. In Section 3 we prove Theorem 1.1 and the associated corollaries. Section 4 is dedicated to the proof of Theorem 1.2. In Section 5, we utilize the group action on $\mathbf{F_D}(N)$ to prove Theorem 1.3 and Theorem 1.4. We end with some concluding remarks in Section 6.

2. Preliminaries

2.1. **Baker's Theorem and its consequence.** We start with the following remarkable theorem of Baker [1].

Theorem 2.1. *Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$. Consequently for any non-zero algebraic numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, the value $\sum_{i=1}^n \beta_i \log \alpha_i$ is either zero or transcendental.*

Lemma 2.1. *Let $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n$, and $\beta_1,\ \beta_2,\ \cdots,\ \beta_m$ be non-zero algebraic numbers. Let K be a number field. Suppose that $\log\beta_1,\ \log\beta_2,\ \cdots,\ \log\beta_m$ are $\mathbb{Q}$-linearly independent. For $c_i \in K$, $1 \le i \le n$, not all zero, set $A := \sum_{i=1}^n c_i \log \alpha_i$. If there exist $d_j \in \overline{\mathbb{Q}}$ for $1 \le j \le m$ such that*

$$A = \sum_{j=1}^{m} d_j \, \log(\beta_j), \tag{11}$$

then $d_j \in K$.

Proof. Let V denote the $\mathbb{Q}$-vector space

$$V = \mathbb{Q}\left\langle \{\log(\alpha_i), \log(\beta_j) \ : \ 1 \le i \le n, 1 \le j \le m\} \right\rangle.$$

The $\mathbb{Q}$-linear independence of $\log \beta_j$'s implies that we can choose a basis of V of the form $\{\log(\beta_j)\}_{j=1}^m \cup \{\log(\gamma_k)\}_{k=1}^{\ell}$, noting that the second set may be empty. Thus, we have

$$\log(\alpha_i) = \sum_{j=1}^{m} r_{j,i}\, \log(\beta_j) + \sum_{k=1}^{\ell} s_{k,i}\, \log(\gamma_k) \qquad r_{j,i},\ s_{k,i} \in \mathbb{Q} \qquad 1 \le i \le n,\ 1 \le j \le m,\ 1 \le k \le \ell.$$

Therefore, by substituting above expressions in the definition of A and comparing it with (11), we get

$$\sum_{j=1}^{m} \log(\beta_j) \left(d_j - \sum_{i=1}^{n} c_i\, r_{j,i} \right) + \sum_{k=1}^{\ell} \log(\gamma_k) \left(\sum_{i=1}^{n} c_i\, s_{k,i} \right) = 0.$$

Using Baker's theorem (Theorem 2.1) and the $\mathbb{Q}$-linear independence of $\log \beta_j$'s and $\log \gamma_k$'s, we deduce that

$$d_j = \sum_{i=1}^{n} c_i\, r_{j,i} \in K \qquad \text{and} \qquad \sum_{i=1}^{n} c_i\, s_{k,i} = 0,$$

which proves the result. □

2.2. **Dirichlet characters and the special value $L(1,\chi)$.** Let q be a positive integer such that $q \mid N$. Let χ be a Dirichlet character $\bmod\, q$. There is a canonical character χ_N mod N, induced from a Dirichlet character χ mod q, defined in the usual way as:

$$\chi_N(n) = \begin{cases} \chi(n) & (n, N) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet series $L(s, \chi_N)$ and $L(s, \chi)$ differ only by some Euler factors, namely,

$$L(s, \chi_N) = L(s, \chi) \prod_{p \mid N} \left(1 - \frac{\chi(p)}{p^s} \right). \tag{12}$$

We now describe a correspondence between Dirichlet characters χ and abelian number fields. Let K be an abelian number field, that is, $K/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(K/\mathbb{Q})$ abelian. By the Kronecker-Weber theorem, $K \subseteq \mathbb{Q}(\zeta_q)$ for some positive integer q. We define the conductor of K

to be the smallest positive integer q such that $K \subseteq \mathbb{Q}(\zeta_q)$.

If $G = G_N = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, then we can associate the character group

$$\widehat{G} = \{\chi \,:\, \chi \text{ is a Dirichlet character modulo } N\}$$

to the cyclotomic field $\mathbb{Q}(\zeta_N)$. As described in [15, Chapter 3], the subfields of $\mathbb{Q}(\zeta_N)$ correspond to the subgroups of $\widehat{G}$ in a bijective way, by mapping the fixed field $\mathbb{Q}(\zeta_N)^H$ to the group of Dirichlet characters $\chi \bmod N$ satisfying the condition $\chi|_H = 1$, where H is a subgroup of G. Also, note that if $\mathbb{Q}(\zeta_q) \subseteq \mathbb{Q}(\zeta_N)$, then there is a natural inclusion of the set of Dirichlet characters modulo q into the set of Dirichlet characters modulo N given by

$$\{\chi \,:\, \chi \bmod q\} \mapsto \{\chi_N \,:\, \chi \bmod q\}.$$

We now describe some results regarding the special value of $L(1, \chi)$ for Dirichlet characters $\chi \bmod N$. Denote by $\tau(\chi)$ the Gauss sum

$$\tau(\chi) = \sum_{a=1}^{N} \chi(a) \zeta_N^{-a}.$$

If χ is a non-principal even primitive Dirichlet character $\bmod N$, then the special value $L(1, \chi)$ is given by :

$$L(1, \chi) = -\frac{\tau(\chi)}{N} \sum_{a=1}^{N-1} \overline{\chi}(a) \log|1 - \zeta_N^a|,$$

whereas for odd primitive Dirichlet characters χ,

$$L(1, \chi) = -\frac{i\pi\tau(\chi)}{N^2} \sum_{a=1}^{N} \overline{\chi}(a) a.$$

We refer the reader to [13, Chapter 2] for details. If the Dirichlet character χ is not primitive, then the determination of $L(1, \chi)$ can be reduced using (12) to the case of primitive characters.

For characters χ associated to an abelian number field K, one can give a precise expression for $L(1, \chi)$, as given in Stark [14] (See the conjecture mentioned in [14, Page 61] and [14, Theorem 1].)

Theorem 2.2. *For a character χ associated with the abelian number field K, we have*

$$L(1, \chi) = \frac{\tau(\overline{\chi})\, 2^a\, \pi^b}{\sqrt{N\,|Disc(K)|}}\, R(\overline{\chi})\, \theta(\overline{\chi}),$$

where $a = \frac{1+\chi(-1)}{2}$, $b = \frac{1-\chi(1)}{2}$, Disc(K) denotes the discriminant of K, $R(\chi)$ is the determinant of an $a \times a$ matrix whose entries are $\overline{\mathbb{Q}}$-linear forms in logarithms of positive units in $\mathcal{O}_K^*$ and $\theta(\chi)$ denotes a certain algebraic number. If $a = 0$, then $R(\chi) = 1$.

We end this section with a result of Murty-Murty [8] and Baker, Birch and Wirsing [2] on the linear independence of $L(1, \chi)$ for non-principal even Dirichlet characters.

Theorem 2.3 (Baker-Birch-Wirsing, Murty-Murty)**.** *The values $L(1, \chi)$ as χ varies over all non-principal even Dirichlet characters* $\bmod N$*, are linearly independent over $\overline{\mathbb{Q}}$. Consequently for any non-zero even algebraic valued function f of Dirichlet type* $\bmod N$*, we have $L(1, f) \neq 0$.*

2.3. **The value $L(1, f)$.** Let K be a number field. We recall some results about the special value $L(1, f)$ for a K-valued arithmetic function f of period N. In our setup, the arithmetic functions will be periodic functions with domain extended to the whole of integers $\mathbb{Z}$.

Let $\zeta(s, x)$ denote the Hurwitz zeta function:

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \qquad 0 < x \leq 1.$$

Then, it is easy to see that

$$L(s, f) = \frac{1}{N^s} \sum_{a=1}^{N} f(a)\, \zeta\left(s, \frac{a}{N}\right). \tag{13}$$

The Hurwitz zeta-function extends analytically to the entire complex plane except for $s = 1$ where it has a simple pole with residue 1. Thus, $L(s, f)$ also extends analytically to the entire complex plane except for a simple pole at $s = 1$ with residue

$$\frac{1}{N} \sum_{a=1}^{N} f(a).$$

Consequently, $L(1, f)$ is finite if and only if

$$\sum_{a=1}^{N} f(a) = 0. \tag{14}$$

Moreover, using the fact that

$$\zeta(s, x) = \frac{1}{s-1} - \psi(x) + O(s-1) \qquad \text{as} \qquad s \to 1^+, \tag{15}$$

it was shown by Murty-Saradha [9] that

$$L(1, f) = -\frac{1}{N} \sum_{a=1}^{N} f(a)\, \psi\left(\frac{a}{N}\right) \tag{16}$$

provided (14) holds. This equation provides the connection between the special value $L(1, f)$ and the N-division values of the digamma function.

As mentioned in the introduction, we have for $f \in \mathbf{F}^{(\mathbf{0})}(N)$,

$$L(1, f) = -\sum_{b=1}^{N-1} \widehat{f}(b) \log\left(1 - \zeta_N^b\right). \tag{17}$$

If f is an odd function of Dirichlet type with period N (that is, $f(-a) = -f(a)$), then the above expression can be simplified to

$$L(1, f) = \frac{i\pi}{2N} \sum_{\substack{a=1 \\ (a,N)=1}}^{N} f(a) \left(\frac{1 + \zeta_N^a}{1 - \zeta_N^a}\right), \tag{18}$$

which can be deduced from equation (2.14) in [13, Chapter 2].

We now recall the results of Baker, Birch and Wirsing which discuss the non-vanishing of $L(s, f)$ at $s = 1$ when f is a periodic function of certain type.

Theorem 2.4 (Baker-Birch-Wirsing)**.** *Let K be a number field and $f : \mathbb{N} \to K$ be a non-zero periodic function of period N satisfying the following conditions:*
(a) *$f(a) = 0$ for all a with $1 < (a, N) < N$*
(b) *$K \cap \mathbb{Q}(\zeta_N) = \mathbb{Q}$*
Then $L(1, f) \neq 0$.

In particular, when f is of Dirichlet type, the Baker-Birch-Wirsing theorem gives us an elegant non-vanishing criterion for $L(1, f)$. For our application of this theorem we restrict ourselves to $K = \mathbb{Q}$ and set $f(N) = 0$.

We now prove the following proposition which gives additional information about the nature of $L(1, f)$. This line of thought is also contained in the proof of [3, Theorem 3.7].

Proposition 2.1. *Let $f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$. Then the value $L(1, f)$ is a $\mathbb{Q}(\zeta_N)$-linear combination of logarithms of units in $\mathbb{Z}[\zeta_N]$.*

Proof. We write $f = f_o + f_e$, where $f_e := \dfrac{f(n) + f(-n)}{2}$, and $f_o := \dfrac{f(n) - f(-n)}{2}$ denote the even and odd components of f respectively. By (18), we can write $L(1, f_o)$ as a $\mathbb{Q}(\zeta_N)$-multiple of $\log(\zeta_N)$. For $L(1, f_e)$, on the one hand, it can be written as a $\overline{\mathbb{Q}}$-linear combination of logarithms of (multiplicatively independent) units in $\mathbb{Z}[\zeta_N]$ by Theorem 2.2 (or [8, Theorem 6]). On the other hand, by (17), $L(1, f_e) \in \mathbb{Q}(\zeta_N) \langle \log(1 - \zeta_N^a) \ : \ 1 \leq a \leq N - 1 \rangle$. Hence, by Lemma 2.1, we obtain that $L(1, f_e)$ is a $\mathbb{Q}(\zeta_N)$-linear combination of logarithms of units in $\mathbb{Z}[\zeta_N]$. Combining the above two observations proves the proposition. $\square$

We conclude this section with a linear independence result. This result is essentially due to the $\overline{\mathbb{Q}}$-linear independence of π and logarithms of positive algebraic numbers. We set f_e and f_o to be the even and odd parts of f defined by the following:

$$f_e(n) := (f(n) + f(-n))/2 \qquad f_o(n) := (f(n) - f(-n))/2$$

Theorem 2.5 (Chatterjee-Murty-Pathak [4])**.** *Suppose $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ is a periodic function of period N. Then, $L(1, f) = 0$ if and only if $L(1, f_e) = 0$ and $L(1, f_o) = 0$.*

2.4. **Definition and Properties of the Induction operator.**

Let g be a function of Dirichlet type modulo q. Since the Dirichlet characters $\chi \bmod q$ form an orthonormal basis for the space of functions $\bmod q$ that are of Dirichlet type, we can write

$$g = \sum_{\chi \bmod q} \langle g, \chi \rangle \, \chi = \sum_{\chi \bmod q} c_\chi(g) \, \chi, \tag{19}$$

where $c_\chi(g) = \langle g, \chi \rangle := \varphi(q)^{-1} \sum_{a=1}^{q} g(a)\overline{\chi}(a)$. We will often suppress the dependence on g and write this as $\sum_\chi c_\chi \chi$.

Recall the definition of the induction operator stated earlier. This is the same operator as defined in [3, Section 4], specialized at $k = 1$. For $q \mid N$ and $g \in \mathbf{F}_D(q)$,

$$\mathrm{Ind}_q^N(g) := \sum_{\chi \bmod q} \frac{c_\chi(g)}{\prod_{p \mid N}(1 - \chi(p)p^{-1})} \ \chi_N,$$

where χ_N denotes the character mod N induced from $\chi \bmod q$. In particular, if $g \in \mathbf{F}_{\mathbf{D}}(q)$, then $\mathrm{Ind}_q^N(g) \in \mathbf{F}_{\mathbf{D}}(N)$. Indeed, it is clear that $\mathrm{Ind}_q^N(g)$ is a function of Dirichlet type with period N

and one only needs to show that it takes rational values. This has been proved in [3, Lemma 4.2].

Moreover, from (12), we have

$$L\left(s, \mathrm{Ind}_q^N(g)\right) = \sum_{\chi \bmod q} \left[\frac{c_\chi(g)}{\prod_{p\mid N} (1-\chi(p)\,p^{-1})} \left(\prod_{p\mid N} \left(1 - \frac{\chi(p)}{p^s}\right)\right) L(s,\chi) \right].$$

Suppose g satisfies the condition $\sum_{a=1}^{q} g(a) = 0$. This translates to $0 = \sum_{a=1}^{q} g(a) = \langle g, \chi_{0,q}\rangle$, where $\chi_{0,q}$ is the principal character modulo q. Therefore, we have $\sum_{a=1}^{N} \mathrm{Ind}_q^N(g)(a) = 0$ and

$$L\left(1, \mathrm{Ind}_q^N(g)\right) = \sum_{\chi \bmod q} c_\chi(g)\, L(1,\chi) = L(1,g). \tag{20}$$

We recall a lemma describing the values of $\mathrm{Ind}_q^N(g)$ in terms of $g \in \mathbf{F_D}(q)$. The proof of this lemma is similar to that described in [3, Lemma 4.4] and we omit the details here.

Lemma 2.2. *Let $g \in \mathbf{F_D}(q)$ and $q \mid N$. For $(n,N)=1$, we have*

$$\mathrm{Ind}_q^N(g)(n) = \sum_{m\in\mathcal{M}_N} \frac{g(mn)}{m}.$$

The periodicity of g will ensure that the values of $\mathrm{Ind}_q^N(g)$ are $\mathbb{Q}$-linear combinations of the values of g. In order to compute the generating function of the Okada constants, we will use the following result [10, Theorem 3.5].

Lemma 2.3. *Let $1 \le a, r \le N$, with $(a,N)=1$, put $t=(r,N)$ and $s=N/t$. We have*

$$t\,A(r,a) = \frac{1}{\varphi(s)} \sum_{\chi \bmod s} \overline{\chi(r/t)}\,\chi(a) \prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\chi(p)}{p}\right)^{-1}.$$

3. Proof of Theorem 1.1

3.1. **The nature of the Okada constants.** We fix a positive integer N. For each prime p dividing N, let $v_p(n)$ denote the highest power of p dividing n. Furthermore, for positive integers $d \mid N$ and primes $p \mid N$, define

$$B_N(d,p) := \frac{\varphi(N)}{\varphi(N/d)} \sum_{\substack{r=1\\ (r,N)=d}}^{N} \sum_{\substack{b\in\mathcal{M}_N\\ b\equiv r \bmod N}} \frac{\nu_p(b)}{b}. \tag{21}$$

The numbers $A(r,n)$ and $B_N(d,p)$ will appear in our proof of Theorem 1.1.

Let $1 \le r,\ n \le N$ with $(n,N)=1$ and p vary over prime divisors of N. In [11], Okada derived an expression for $A(r,n)$'s to conclude their rationality. A different proof of this was given in [10, Section 3]. We record this in the statement below.

Lemma 3.1. *For $1 \le r,\ n < N$ with $(n,N)=1$, the numbers $A(r,n)$ are rational.*

In the lemma below, we show that the numbers $B_N(d,p)$ are also rational.

Lemma 3.2. *For $d \mid N$, the numbers $B_N(d,p)$ are rational. More precisely,*

$$B_N(d,p) = \begin{cases} \nu_p(d) & \text{if } p \mid (N/d) \\ \nu_p(d) + \dfrac{1}{p-1} & \text{otherwise.} \end{cases}$$

Proof. Using the definition (21) of $B_N(d,p)$, we have

$$B_N(d,p) = \frac{\varphi(N)}{\varphi(N/d)} \sum_{\substack{r=1 \\ (r,N)=d}}^{N} \sum_{\substack{b \in \mathcal{M}_N \\ b \equiv r \bmod N}} \frac{\nu_p(b)}{b} = \frac{\varphi(N)}{\varphi(N/d)} \sum_{\substack{b \in \mathcal{M}_N \\ (b,N)=d}} \frac{\nu_p(b)}{b}. \tag{22}$$

We obtain the last equality by noting that $(b/d, N/d) = 1$. Now for $b \in \mathcal{M}_N$ divisible by d, we write $b = b'd$ with $b' \in \mathcal{M}_N$. Therefore, we have

$$B_N(d,p) = \frac{\varphi(N)}{\varphi(N/d)} \sum_{\substack{b' \in \mathcal{M}_N \\ (b',N/d)=1}} \frac{\nu_p(b'd)}{b'd} = \frac{\varphi(N)}{d\varphi(N/d)} \left(\nu_p(d) \sum_{\substack{b' \in \mathcal{M}_N \\ (b',N/d)=1}} \frac{1}{b'} + \sum_{\substack{b' \in \mathcal{M}_N \\ (b',N/d)=1}} \frac{\nu_p(b')}{b'} \right). \tag{23}$$

The set of elements in $\mathcal{M}_N$ coprime to N/d also form a monoid. Therefore, by substituting $s = 1$ in (4) we obtain

$$\sum_{\substack{b' \in \mathcal{M}_N \\ (b',N/d)=1}} \frac{1}{b'} = \prod_{\substack{\ell \mid N \\ \ell \nmid (N/d)}} \left(1 - \frac{1}{\ell}\right)^{-1} = \frac{d\,\varphi(N/d)}{\varphi(N)},$$

where ℓ runs over the prime numbers dividing N but not dividing N/d. It remains to evaluate the second summand in (23) and this is non-zero when we have $p \nmid (N/d)$.

In this particular case, we have

$$\sum_{\substack{b' \in \mathcal{M}_N \\ (b',N/d)=1}} \frac{\nu_p(b')}{b'} = \sum_{j=1}^{\infty} \frac{j}{p^j} \sum_{\substack{b_1 \in \mathcal{M}_N \\ (b_1,pN/d)=1}} \frac{1}{b_1} = \frac{1}{p(1-1/p)^2} \prod_{\substack{\ell \mid N \\ \ell \nmid pN/d}} \left(1 - \frac{1}{\ell}\right)^{-1} = \left(\frac{d}{p-1}\right) \frac{\varphi(N/d)}{\varphi(N)},$$

where in the second sum, we set $b' = p^j b_1$ with b_1 coprime to p. Inserting this in (23), we obtain the lemma. ◻

As an immediate consequence of the above computation, we get

$$\sum_{p \mid N} B_N(d,p) \log p = \log d + \sum_{\substack{p \mid N \\ p \nmid (N/d)}} \frac{\log p}{p-1}. \tag{24}$$

Another observation which is useful in the proofs of the forthcoming theorems is as follows. For $1 \le r \le N$, let $\rho_r(f) := \sum_{\substack{m=1 \\ (m,N)=1}}^{N} f(rm)$ and for $(r,N) = d$, set $P_d(f) := \sum_{\substack{m=1 \\ (m,N/d)=1}}^{N/d} f(dm)$. Then we have the following relation.

$$\rho_r(f) = \frac{\varphi(N)}{\varphi(N/d)} P_d(f). \tag{25}$$

3.2. **Proof of Theorem 1.1.** Towards the proof of Theorem 1.1, we first establish the following general result.

Theorem 3.1. *Let $K \subseteq \overline{\mathbb{Q}}$ be a field and $f : \mathbb{Z} \to K$ be an arithmetic function, periodic with period $N \geq 2$, satisfying* (14)*. Then,*

$$L(1,f) = \frac{-1}{N}\left(\sum_{\substack{n=1,\\(n,N)=1}}^{N} \psi\left(\frac{n}{N}\right)\left(\sum_{r=1}^{N} f(rn)\, A(r,1)\right) + \sum_{\substack{p\mid N,\\ p \text{ prime}}} \log p \left(\sum_{d\mid N} P_d(f)\, B_N(d,p)\right)\right).$$

Moreover, the coefficients of $\psi(n/N)$ and $\log p$ in the above expression lie in K.

Proof. From the definition of $L(s,f)$, for $\mathrm{Re}(s) > 1$, we have

$$L(s,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{b\in\mathcal{M}_N} \frac{1}{b^s} \sum_{\substack{n=1,\\(n,N)=1}}^{\infty} \frac{f(bn)}{n^s} = \sum_{b\in\mathcal{M}_N} \frac{L(s,f_b)}{b^s},$$

where

$$f_b(m) := \begin{cases} f(bm) & \text{if } (m,N)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\rho_b := \sum_{m=1}^{N} f_b(m)$. Then, using (13) and the Laurent expansion of the Hurwitz zeta function (15) at $s=1$, we obtain that

$$\begin{aligned} L(s,f) &= \frac{1}{N^s\,(s-1)}\left(\sum_{b\in\mathcal{M}_N} \frac{\rho_b}{b^s}\right) - \frac{1}{N^s} \sum_{b\in\mathcal{M}_N} \frac{1}{b^s} \sum_{\substack{n=1,\\(n,N)=1}}^{N} f_b(n)\, \psi\left(\frac{n}{N}\right) + O(s-1)\\ &= \frac{1}{N^s\,(s-1)}\left(\sum_{b\in\mathcal{M}_N} \frac{\rho_b}{b^s}\right) - \frac{1}{N^s} \sum_{\substack{n=1,\\(n,N)=1}}^{N} \psi\left(\frac{n}{N}\right)\left(\sum_{b\in\mathcal{M}_N} \frac{f_b(n)}{b^s}\right) + O(s-1). \end{aligned}$$

Since $L(s,f)$ is entire, we deduce that

$$\lim_{s\to1^+} \sum_{b\in\mathcal{M}_N} \frac{\rho_b}{b^s} = 0.$$

The absolute and uniform convergence of the associated series can be seen from (4). Thus by L'Hôpital's rule, we obtain that

$$\lim_{s\to1^+} \frac{1}{(s-1)} \sum_{b\in\mathcal{M}_N} \frac{\rho_b}{b^s} = -\lim_{s\to1^+} \sum_{b\in\mathcal{M}_N} \frac{(\log b)\,\rho_b}{b^s} = -\sum_{b\in\mathcal{M}_N} \frac{(\log b)\rho_b}{b}.$$

The last sum in the above equation can be re-written as

$$\sum_{b\in\mathcal{M}_N} \frac{(\log b)\rho_b}{b} = \sum_{b\in\mathcal{M}_N} \frac{\rho_b}{b}\left(\sum_{p\mid b} v_p(b)\, \log p\right) = \sum_{\substack{p\mid N,\\ p \text{ prime}}} \log p \left(\sum_{\substack{b\in\mathcal{M}_N,\\ p\mid b}} \frac{\nu_p(b)\,\rho_b}{b}\right),$$

where $\nu_p(b)$ is the power to which the prime p divides b and we have interchanged summations in the penultimate step. Therefore, we will be done once we have shown that the terms

$$\mathcal{S}_f(n) := \sum_{b\in\mathcal{M}_N} \frac{f_b(n)}{b} \qquad \text{and} \qquad T_f(p) := \sum_{b\in\mathcal{M}_N} \frac{\nu_p(b)\,\rho_b}{b}$$

are equal to the coefficients of $\psi(n/N)$ and $\log p$ respectively in the statement of the theorem, and that these terms lie in K . Note that for $(n, N) = 1$, $f_b(n) = f_c(n)$ if $b \equiv c \bmod N$ and hence, $\rho_b = \rho_c$ if $b \equiv c \bmod N$. Therefore, $\mathcal{S}_f(n)$ and $T_f(p)$ can be written as

$$\sum_{b \in \mathcal{M}_N} \frac{f_b(n)}{b} = \sum_{r=1}^{N} f_r(n) \sum_{\substack{b \in \mathcal{M}_N \\ b \equiv r \bmod N}} \frac{1}{b} = \sum_{r=1}^{N} f_r(n)\, A(r, 1),$$

and

$$\sum_{b \in \mathcal{M}_N} \frac{\nu_p(b)\, \rho_b}{b} = \sum_{r=1}^{N} \rho_r \sum_{\substack{b \in \mathcal{M}_N \\ b \equiv r \bmod N}} \frac{\nu_p(b)}{b} = \sum_{d|N} \rho_d \sum_{\substack{r=1 \\ (r,N)=d}}^{N} \sum_{\substack{b \in \mathcal{M}_N \\ b \equiv r \bmod N}} \frac{\nu_p(b)}{b} = \sum_{d|N} P_d(f)\, B_N(d, p).$$

To obtain the last equality, we used (25) and (22). By Lemma 3.1 and Lemma 3.2, we deduce that $\mathcal{S}_f(n)$ and $T_f(p) \in K$. This proves the theorem. □

Now to complete the proof of the main theorem, we make a judicious choice of the periodic function f.

Proof of Theorem 1.1. Let $1 \le \delta < N$ with $\gcd(\delta, N) = d$. If $d = 1$, that is, $(\delta, N) = 1$, then $A(\delta, n) = 0$ for $(n, N) = 1$ unless $n = \delta$, in which case, $A(\delta, \delta) = 1$. Moreover, the number $B_N(1, p) = 0$ for all $p \mid N$. These observations follow immediately from the definitions of $A(r, n)$'s and $B_N(r, p)$'s. Thus, the theorem is true when $d = 1$.

Now suppose that $d > 1$. We can assume that $\delta \not\equiv 1 \bmod N$. Define f to be the periodic arithmetic function such that

$$f(n) = \begin{cases} 1 & \text{if } n \equiv \delta \bmod N \\ -1 & \text{if } n \equiv 1 \bmod N \\ 0 & \text{otherwise.} \end{cases}$$

We will use Theorem 3.1 to calculate $L(1, f)$. Firstly, for $(n, N) = 1$ and $1 \le r \le N$, $f_r(n) = f(rn)$ is non-zero only if $rn \equiv \delta \bmod N$ or if $rn \equiv 1 \bmod N$. Thus, $f_r(n) = -1$ if $r \equiv n^{-1} \bmod N$ and $A(r, 1) \ne 0$ if and only if $r = 1$ when $(r, N) = 1$. Thus, for $n > 1$ and coprime to N,

$$\mathcal{S}_f(n) := \sum_{r=1}^{N} f_r(n)\, A(r, 1) = \sum_{\substack{r=1 \\ rn \equiv \delta \bmod N}}^{N} A(r, 1) = \sum_{\substack{r=1 \\ rn \equiv \delta \bmod N}}^{N} \sum_{\substack{b \in \mathcal{M}_N \\ b \equiv r \bmod N}} \frac{1}{b} = \sum_{b \in \mathcal{M}_N} \frac{1}{b} \sum_{\substack{r=1 \\ rn \equiv \delta \bmod N \\ b \equiv r \bmod N}}^{N} 1,$$

where the inner sum is 1 if $r \equiv b \bmod N$ and satisfies $bn \equiv \delta \bmod N$. In other words,

$$\mathcal{S}_f(n) = \sum_{\substack{b \in \mathcal{M}_N \\ bn \equiv \delta \bmod N}} \frac{1}{b} = A(\delta, n).$$

If $n = 1$, we have an extra term as $f_1(1) = -1$ and $A(1, 1) = 1$. Hence,

$$\mathcal{S}_f(1) = A(\delta, 1) - 1.$$

To determine the coefficients of $\log p$, we note that for divisors e of N, we have $P_e(f) = 0$ for $e \ne d, 1$. By substituting these values in Theorem 3.1, we obtain

$$\sum_{e|N} P_e(f) B_N(e, p) = f(1) B_N(1, p) + f(\delta) B_N(d, p) = B_N(d, p).$$

From (16) we obtain $\psi(\delta/N) = \psi(1/N) - N L(1, f)$. Using now Theorem 3.1 and (24) with the quantities calculated above, yields the theorem.

□

Remark 3.1. *Theorem 1.1 can be reinterpreted as follows. Let* d, N *be positive integers such that* $d \mid N$. *Let* $a_i \in \mathbb{Q}$ *for* $1 \le i \le d, (i,d) = 1$ *and let*

$$\alpha := \sum_{\substack{i=1 \\ (i,d)=1}}^{d} a_i \, \psi\left(\frac{i}{d}\right)$$

be an element of $\mathscr{D}_P(d)$, *the* $\mathbb{Q}$*-vector space spanned by primitive* d*-division values. Set* ρ_α *to be the sum* $\sum\limits_{\substack{i=1 \\ (i,d)=1}}^{d} a_i$. *Then Theorem 1.1 proves that*

$$\alpha - \left[\rho_\alpha \left(\log \frac{N}{d} + \sum_{\substack{p \mid N \\ p \nmid d}} \frac{\log p}{p-1}\right)\right] \in \mathscr{D}_P(N).$$

In particular, if $\rho_\alpha = 0$, *then* $\alpha \in \mathscr{D}_P(N)$.

3.3. **Towards the Murty-Saradha conjecture.** We begin with an important observation regarding linear relations among primitive division values of the digamma function. This is necessary in the proof of Corollary 1.1.

Lemma 3.3. *Let* $N > 1$ *be a positive integer, and suppose there exists rational numbers* a_i $1 \le i \le N$, $(i,N) = 1$ *not all zero such that*

$$\sum_{\substack{i=1 \\ (i,N)=1}}^{N} a_i \, \psi\left(\frac{i}{N}\right) = 0.$$

Then we have $\sum\limits_{\substack{i=1 \\ (i,N)=1}}^{N} a_i \neq 0$.

Proof. Assume the contrary, that is, suppose there are rational numbers a_i satisfying

$$\sum_{\substack{i=1 \\ (i,N)=1}}^{N} a_i \, \psi\left(\frac{i}{N}\right) = 0, \qquad \text{and} \qquad \sum_{\substack{i=1 \\ (i,N)=1}}^{N} a_i = 0.$$

We can now construct a $\mathbb{Q}$-valued rational function $f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$ by setting $f(i) = a_i$ for all $1 \le i \le N$ and $(i,N) = 1$. From the hypothesis on a_i, f satisfies (14) and therefore by (16), $L(1,f) = 0$. By Theorem 2.4, this implies that $f \equiv 0$, and hence $a_i = 0$ for all i. This proves the result. □

We also recall [3, Lemma 3.6] which is a direct consequence of Baker's theorem (Theorem 2.1) and the fact that that prime ideals do not contain units.

Lemma 3.4. *Let* $\mathbb{F}$ *be a number field and* $u_1, u_2, \cdots, u_n \in \mathcal{O}_{\mathbb{F}}^*$ *be units. Let* $p_1, p_2, \cdots, p_m$ *be rational primes. Then*

$$\overline{\mathbb{Q}} \langle \log u_1, \log u_2, \cdots, \log u_n \rangle \cap \overline{\mathbb{Q}} \langle \log p_1, \log p_2, \cdots, \log p_m \rangle = \{0\}.$$

We are now ready to prove that there can be at most one possible exception to the Murty-Saradha conjecture regarding the dimension of $\mathscr{D}_P(N)$.

Proof of Corollary 1.1. Suppose for two distinct natural numbers r and s, we have $\dim_{\mathbb{Q}} \mathscr{D}_P(r) < \varphi(r)$ and $\dim_{\mathbb{Q}} \mathscr{D}_P(s) < \varphi(s)$. Then for $1 \leq i \leq r$ with $(i,r)=1$ and $1 \leq j \leq s$ with $(j,s)=1$, there are relations with $c_i, d_j \in \mathbb{Q}$ and not all c_i and not all d_j being zero such that,

$$\alpha_r := \sum_{\substack{i=1 \\ (i,r)=1}}^{r} c_i\, \psi(i/r) = 0 \qquad \text{and} \qquad \alpha_s := \sum_{\substack{j=1 \\ (j,s)=1}}^{s} d_j\, \psi(j/s) = 0.$$

By Lemma 3.3, we have that $\rho_{\alpha_r} := \sum_{\substack{i=1 \\ (i,r)=1}}^{r} c_i \neq 0$ and $\rho_{\alpha_s} := \sum_{\substack{i=1 \\ (i,s)=1}}^{s} d_i \neq 0$. Let t be a prime not dividing rs and set $N = rst$. Since $\alpha_r = \alpha_s = 0$ and $\rho_{\alpha_r}, \rho_{\alpha_s} \in \mathbb{Q}$, Remark 3.1 implies that

$$\mathcal{R}_j := \log(N/j) + \sum_{\substack{p \mid (N/j) \\ p \nmid j}} \frac{\log p}{p-1} \quad \in \mathscr{D}_P(N), \qquad \text{for} \qquad j = r,\, s. \tag{26}$$

Now, we claim that $\mathcal{R}_r = \mathcal{R}_s$. Indeed, let

$$\mathcal{R}_j = \sum_{\substack{k=1 \\ (k,N)=1}}^{N} g_j(k)\, \psi\left(\frac{k}{N}\right) \in \mathscr{D}_P(N) \qquad \text{for} \qquad j = r,\, s.$$

For $j = r$ and s, define $\widetilde{\rho_j} := \sum_{(k,N)=1} g_j(k) \in \mathbb{Q}$. If $\widetilde{\rho_j} = 0$ for $j = r$ or s, then $\mathcal{R}_j = L(1, g_j)$ with $g_j \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$. Proposition 2.1 would then imply that $\mathcal{R}_j$ is a $\overline{\mathbb{Q}}$-linear combination of units in $\mathbb{Z}[\zeta_N]$. However, by expression (26), $\mathcal{R}_j$ is a $\mathbb{Q}$-linear combination of logarithm of primes dividing N. Thus, Lemma 3.4 gives that $\mathcal{R}_j = 0$, which is a contradiction to (26). Hence, $\widetilde{\rho_j} \neq 0$ for $j = r$ and s. Now observe that

$$\widetilde{\rho_r}\mathcal{R}_s - \widetilde{\rho_s}\mathcal{R}_r = \sum_{\substack{k=1 \\ (k,N)=1}}^{N} \left(\widetilde{\rho_r}\, g_r(k) - \widetilde{\rho_s}\, g_s(k)\right)\, \psi\left(\frac{k}{N}\right).$$

Since the sum of coefficients of $\psi(k/N)$ in the expression above is zero, we obtain that

$$\widetilde{\rho_r}\mathcal{R}_s - \widetilde{\rho_s}\mathcal{R}_r \in \mathbb{Q}\left\langle L(1,f) \;:\; f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N) \right\rangle.$$

Once again, $\widetilde{\rho_r}\mathcal{R}_s - \widetilde{\rho_s}\mathcal{R}_r$ is a $\mathbb{Q}$-linear combination of logarithm of primes dividing N by expression (26). On the other hand, Proposition 2.1 implies that $\widetilde{\rho_r}\mathcal{R}_s - \widetilde{\rho_s}\mathcal{R}_r$ is a $\overline{\mathbb{Q}}$-linear combination of logarithms of units in $\mathbb{Z}[\zeta_N]$. Hence, Lemma 3.4 gives $\widetilde{\rho_r}\mathcal{R}_s - \widetilde{\rho_s}\mathcal{R}_r = 0$.

$$0 = \widetilde{\rho_s}\mathcal{R}_r - \widetilde{\rho_r}\mathcal{R}_s = \frac{t\,(\widetilde{\rho_s} - \widetilde{\rho_r})}{t-1}\, \log t + \widetilde{\rho_s}\left(\log r + \sum_{\substack{p \mid r \\ p \nmid s}} \frac{\log p}{p-1}\right) - \widetilde{\rho_r}\left(\log s + \sum_{\substack{p \mid s \\ p \nmid r}} \frac{\log p}{p-1}\right).$$

Since$(t, rs) = 1$, we obtain $\widetilde{\rho_r} = \widetilde{\rho_s}$, and therefore $\mathcal{R}_r = \mathcal{R}_s$. Thus,

$$\log r + \sum_{\substack{p \mid r \\ p \nmid s}} \frac{\log p}{p-1} = \log s + \sum_{\substack{p \mid s \\ p \nmid r}} \frac{\log p}{p-1}.$$

Adding $\sum_{p|(s,r)} \frac{\log p}{p-1}$ to both sides of the above equation gives

$$\log r + \sum_{p|r} \frac{\log p}{p-1} = \log s + \sum_{p|s} \frac{\log p}{p-1}.$$

From here, it follows that $r = s$. This completes the proof. $\square$

Proof of Corollary 1.2. Recall that $\psi(1-x) - \psi(x) = \pi \cot(\pi x)$. We apply Theorem 1.1 to $\psi\left(\frac{\delta}{N}\right)$ and $\psi\left(\frac{N-\delta}{N}\right)$. This gives

$$\psi\left(\frac{\delta}{N}\right) = \sum_{\substack{n=1\\(n,N)=1}}^{N} A(\delta, n)\ \psi\left(\frac{n}{N}\right) + \log d + \sum_{\substack{p|N,\\ p\nmid(N/d)\\ p \text{ prime}}} \frac{\log p}{p-1}.$$

$$\psi\left(\frac{N-\delta}{N}\right) = \sum_{\substack{n=1\\(n,N)=1}}^{N} A(N-\delta, n)\ \psi\left(\frac{n}{N}\right) + \log d + \sum_{\substack{p|N,\\ p\nmid(N/d)\\ p \text{ prime}}} \frac{\log p}{p-1}.$$

Therefore,

$$\pi \cot \frac{\pi\delta}{N} = \sum_{\substack{n=1\\(n,N)=1}}^{N} \left(A(N-\delta, n) - A(\delta, n)\right)\ \psi\left(\frac{n}{N}\right).$$

By (3), we have $A(N-\delta, n) = A(\delta, N-n)$. On substituting this in the above equation, we have

$$\begin{aligned}\pi \cot \frac{\pi\delta}{N} &= \sum_{\substack{n=1\\(n,N)=1}}^{N} A(\delta, N-n)\ \psi\left(\frac{n}{N}\right) - \sum_{\substack{n=1\\(n,N)=1}}^{N} A(\delta, n)\ \psi\left(\frac{n}{N}\right)\\ &= \sum_{\substack{n=1\\(n,N)=1}}^{N} A(\delta, n)\ \left(\psi\left(1-\frac{n}{N}\right) - \psi\left(\frac{n}{N}\right)\right).\end{aligned}$$

Applying the reflection identity again we get the desired result. $\square$

4. **Okada constants and elements in the ring $\mathbb{Q}[\mathcal{G}_{N/q}]$**

In this section, we establish a generating function for the Okada constants as elements in the group ring $\overline{\mathbb{Q}}[\mathcal{G}_{N/q}]$ and prove Theorem 1.2. Here r and N are natural numbers greater than 1, $q = (r, N)$, $\mathcal{G}_{N/q} \cong (\mathbb{Z}/(N/q)\mathbb{Z})^*$, and the elements of $\mathcal{G}_{N/q}$ are denoted by σ_c where $(c, N/q) = 1$. For our setup, we emphasize that $\mathcal{G}_{N/q}$ acts by left multiplication on $\overline{\mathbb{Q}}[\mathcal{G}_{N/q}]$, that is

$$\sigma_c \cdot \sum_{\substack{b=1\\(b,N/q)=1}}^{N/q} \beta_b\, \sigma_b = \sum_{\substack{b=1\\(b,N/q)=1}}^{N/q} \beta_b\ (\sigma_c \cdot \sigma_b) \qquad \text{for} \qquad \beta_b \in \overline{\mathbb{Q}}.$$

Fix any natural number n with $(n, N) = 1$.

Proof of Theorem 1.2. We set $s = N/q$. By using Lemma 2.3, we see that the generating function is given by

$$\begin{aligned}\sum_{\substack{b=1\\(b,s)=1}}^{s} A(qb,n)\,\sigma_b &= \sum_{\substack{b=1\\(b,s)=1}}^{s} \frac{1}{q\varphi(s)} \sum_{\chi \bmod s} \overline{\chi(b)}\,\chi(n) \prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \sigma_b \\ &= \frac{\sigma_n}{q} \sum_{\chi \bmod s} \prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \epsilon_\chi,\end{aligned}$$

where the idempotent ϵ_χ is given by:

$$\epsilon_\chi = \frac{1}{\varphi(s)} \sum_{\substack{b=1\\(b,s)=1}}^{s} \overline{\chi(b)}\,\sigma_b.$$

Using the fact that $\sigma_n\,\epsilon_\chi = \chi(n)\,\epsilon_\chi$ (See [15, Chapter 6]), we note that

$$\begin{aligned}\prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\sigma_p}{p}\right) \sum_{\substack{b=1\\(b,s)=1}}^{s} A(qb,n)\,\sigma_b &= \frac{1}{q} \prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\sigma_p}{p}\right) \sigma_n \sum_{\chi \bmod s} \prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \epsilon_\chi \\ &= \frac{\sigma_n}{q} \sum_{\chi \bmod s} \prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \prod_{\substack{p\mid N\\ p\nmid s}} \left(1 - \frac{\sigma_p}{p}\right) \epsilon_\chi \\ &= \frac{\sigma_n}{q} \sum_{\chi \bmod s} \epsilon_\chi = \frac{\sigma_n}{q}.\end{aligned}$$

This proves the result. □

5. Galois Correspondence and $L(1,f)$

Let K be a number field and f be an arithmetic function, periodic with period q taking values in K. It is clear from (17), that $L(1,f)$ is a linear form in logarithms of algebraic numbers in $\mathbb{Q}(\zeta_q)$ with coefficients in $K(\zeta_q)$. In this section, we focus on the relationship between the period of f and the field of coefficients of the linear form in logarithm $L(1,f)$. This is captured in Theorem 1.3. This is then used to determine the image of $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(q)$ under the induction operator Ind_q^N.

Recall that if $G = G_N = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, the element $\sigma_n \in G$ is defined by $\sigma_n(\zeta_N) = \zeta_N^n$; where $(n,N) = 1$. Also, if $f \in \mathbf{F}_D(N)$, then $\sigma_n^{-1}(f)(m) = f(mn)$ for all integers m. Moreover for a subgroup H of G, we define

$$\begin{aligned}\mathbf{F}_{\mathbf{D}}(N)^H &:= \{f \in \mathbf{F}_{\mathbf{D}}(N) : \sigma(f) = f \text{ for all } \sigma \in H\} \\ \mathbf{F}_{\mathbf{D}}(N)_H &:= \{f \in \mathbf{F}_{\mathbf{D}}(N) : \langle f, \chi\rangle = 0 \quad \text{for all } \chi \bmod N \text{ such that } \chi|_H \neq \{1\}\}.\end{aligned}$$

With this notation, we have the following equality of subspaces of $\mathbf{F}_{\mathbf{D}}(N)$.

Proposition 5.1. *Let H be subgroup of G. Then*

$$\mathbf{F}_{\mathbf{D}}(N)^H = \mathbf{F}_{\mathbf{D}}(N)_H \qquad \textit{and} \qquad \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H = \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)_H.$$

Proof. Let $f \in \mathbf{F_D}(N)$. Then $f = \sum_{\chi \bmod N} \langle f, \chi \rangle \, \chi$ and for all natural numbers m,

$$f(nm) = \sum_{\chi \bmod N} \langle f, \chi \rangle \, \chi(n) \, \chi(m).$$

Hence, the arithmetic function $\sigma_n^{-1}(f)$ has the character decomposition

$$\sigma_n^{-1}(f) = \sum_{\chi \bmod N} \langle f, \chi \rangle \, \chi(n) \, \chi. \tag{27}$$

Now, if $\langle f, \chi \rangle = 0$ for $\chi \bmod N$ such that $\chi|_H \neq \{1\}$ and $\sigma_n \in H$, then

$$\sigma_n^{-1}(f) = \sum_{\substack{\chi \bmod N, \\ \chi|_H = 1}} \langle f, \chi \rangle \, \chi(n) \, \chi = \sum_{\substack{\chi \bmod N, \\ \chi|_H = 1}} \langle f, \chi \rangle \, \chi = f,$$

which implies that $f \in \mathbf{F_D}(N)^H$.

Conversely, suppose that $f \in \mathbf{F_D}(N)^H$. By orthogonality relations for characters, we have

$$\frac{1}{|H|} \sum_{\sigma_n \in H} \chi(n) = \begin{cases} 1 & \text{if } \chi|_H = \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

Using this and (27), we get

$$f = \frac{1}{|H|} \sum_{\sigma_n \in H} \sigma_n^{-1}(f) = \sum_{\chi \bmod N} \langle f, \chi \rangle \left(\frac{1}{|H|} \sum_{\sigma_n \in H} \chi(n) \right) \chi. \tag{28}$$

This proves the claim. The same argument also establishes the equality for $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$. $\square$

Corollary 5.1. *Let q and N be natural numbers such that $q \mid N$. Let $H = \mathrm{Gal}\,(\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_q))$. Then*

$$\mathrm{Ind}_q^N \Big(\mathbf{F_D}(q) \Big) = \mathbf{F_D}(N)^H.$$

Proof. If $f = \mathrm{Ind}_q^N(g)$ for some $g \in \mathbf{F_D}(q)$, then Definition 1.1 and (27) together imply that $f \in \mathbf{F_D}(N)^H$. Now, suppose $f \in \mathbf{F_D}(N)^H$. By Proposition 5.1, $\langle f, \chi \rangle = 0$ for all characters χ such that $\chi|_H \neq \{1\}$. Therefore, f is supported on the characters χ for which $\chi|_H = \{1\}$ and these are precisely the characters $\chi \bmod q$. Therefore,

$$f = \sum_{\chi \bmod q} \langle f, \chi_N \rangle \, \chi_N.$$

Construct a periodic function g of period q, supported on the coprime residue classes as

$$g = \sum_{\chi \bmod q} \langle f, \chi_N \rangle \left[\prod_{p \mid N} \left(1 - \frac{\chi(p)}{p} \right) \right] \chi.$$

We claim that $g \in \mathbf{F_D}(q)$. By Definition 1.1, it suffices to check that g is rational valued. This follows from the invariance of the above expression of g under Galois automorphisms. Let $\omega \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For $(n,q)=1$, we have

$$\begin{aligned}\omega(g(n)) &= \sum_{\chi \bmod q} \omega\left(\langle f, \chi_N\rangle \left[\prod_{p|N}\left(1-\frac{\chi(p)}{p}\right)\right]\chi(n)\right)\\ &= \sum_{\chi \bmod q} \langle f, \omega(\chi)_N\rangle \left[\prod_{p|N}\left(1-\frac{\omega(\chi(p))}{p}\right)\right]\omega(\chi(n))\\ &= g(n),\end{aligned}$$

where we get the last equality by noting that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes the group of Dirichlet characters mod q. Hence the rationality of g follows from the rationality of f. □

In order to prove Theorem 1.3, we first establish equality of certain associated vector spaces. This is because it is difficult to prove Theorem 1.3 directly. In greater detail, given $f \in \mathbf{F}_{\mathbf{D}}^{\mathbf{0}}(N)$ such that $L(1,f) \in K^+ \langle \log|u| \,:\, u \in O_K^*\rangle$, it is difficult to show that $f \in \mathbf{F}_{\mathbf{D}}^{\mathbf{0}}(N)^H$ owing to the non-explicit description of $L(1,f)$ as linear forms of logarithm of units in $\mathbb{Q}(\zeta_N)^+$.

Proposition 5.2. *Let K be a number field contained in $\mathbb{Q}(\zeta_N)$ and let $H = Gal(\mathbb{Q}(\zeta_N)/K)$. Let $\tau_c : \mathbb{Q}(\zeta_N) \to \mathbb{Q}(\zeta_N)$ denote complex conjugation. Define*

$$K^- := \{z \in K \,:\, \tau_c(z) = -z\},$$

and $K^+ := K \cap \mathbb{R}$. Then we have the following equality of vector spaces.

(i) $\overline{\mathbb{Q}}\left\langle\left\{L(1,f) \,:\, f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H \text{ and } f \text{ is even}\right\}\right\rangle = \overline{\mathbb{Q}}\left\langle\left\{\log|u| \,:\, u \in \mathcal{O}_{K^+}^*\right\}\right\rangle.$

(ii) $\left\{L(1,f) \,:\, f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H \text{ and } f \text{ is odd}\right\} = i\pi \cdot K^-.$

Proof. In each case, we shall show that $L(1,f)$ lies in the vector space on the right hand side using the condition imposed on f. To prove the equality of vector spaces, we compare the dimensions ($\overline{\mathbb{Q}}$-dimension for (i) and $\mathbb{Q}$-dimension for (ii)) of the spaces spanned by $L(1,f)$ using the linear independence results mentioned in Section 2, and the dimension of the space spanned by logarithms of units using Dirichlet's unit theorem.

- Proof of part (i): Define H_1 to be the group generated by H and the complex conjugation $\tau_c \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. Then we have,

$$\begin{aligned}&\overline{\mathbb{Q}}\left\langle\left\{f \,:\, f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H \text{ and } f \text{ is even}\right\}\right\rangle = \overline{\mathbb{Q}}\left\langle\left\{f \,:\, f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^{H_1}\right\}\right\rangle\\ =&\overline{\mathbb{Q}}\left\langle\{\chi \bmod N \,:\, \chi \neq \chi_{0,N},\, \chi|_H = \{1\} \text{ and } \chi \text{ is even}\}\right\rangle \qquad (29)\\ =&\overline{\mathbb{Q}}\left\langle\{\chi \bmod N \,:\, \chi|_{H_1} = \{1\} \text{ and } \chi \neq \chi_{0,N}\}\right\rangle, \qquad (30)\end{aligned}$$

where the last two equalities follow from Proposition 5.1. Therefore, it suffices to show that for all non-principal characters χ mod N such that $\chi|_{H_1} = \{1\}$, we have $L(1,\chi) \in \overline{\mathbb{Q}}\langle\{\log|u| \,:\, u \in \mathcal{O}_K^*\}\rangle$. Now, let χ be a non-trivial even character associated to the field K. Hence, χ is associated to the maximal real subfield K^+ of K. By Stark's result, Theorem 2.2, we can write

$$L(1,\chi) = \sum_{\sigma \in \mathrm{Gal}(K^+/\mathbb{Q})} a_\chi(\sigma)\, \log|\alpha_\sigma|,$$

for some $a_\chi(\sigma) \in \overline{\mathbb{Q}}$ and some $\alpha_\sigma \in \mathcal{O}_{K^+}^*$. Thus, we obtain the inclusion of vector spaces

$$\overline{\mathbb{Q}}\left\langle\left\{L(1,f) \,:\, f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H \text{ and } f \text{ is even}\right\}\right\rangle \subseteq \overline{\mathbb{Q}}\left\langle\{\log|u| \,:\, u \in \mathcal{O}_{K^+}^*\}\right\rangle.$$

We now compute the $\overline{\mathbb{Q}}$-dimension of the vector space generated by the $L(1,f)$ values. By Theorem 2.3, (29) there exists no non-zero even algebraic valued Dirichlet type function f of period N such that $L(1,f)=0$. On combining this fact with (30), we have

$$\begin{aligned}
&\dim_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}\left\langle\left\{L(1,f)\ :\ f\in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^{H_1}\right\}\right\rangle\\
&=\dim_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}\left\langle\left\{f\ :\ f\in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^{H_1}\right\}\right\rangle\\
&=\dim_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}\left\langle\{\chi \bmod N\ :\ \chi|_{H_1}=\{1\}\text{ and }\chi\neq\chi_{0,N}\}\right\rangle\\
&=\frac{\varphi(N)}{|H_1|}-1,
\end{aligned}$$

by the linear independence of Dirichlet characters. On the other hand, by Dirichlet's unit theorem, the rank of the non-torsion part of the unit group $\mathcal{O}^*_{K^+}$ is $[K^+:\mathbb{Q}]-1$. Using a fundamental system of units for $\mathcal{O}^*_{K^+}$ and applying Baker's theorem, we have

$$\dim_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}\left\langle\{\log|u|\ :\ u\in\mathcal{O}^*_{K^+}\}\right\rangle=[K^+:\mathbb{Q}]-1=\frac{\varphi(N)}{|H_1|}-1.$$

This proves the equality of $\overline{\mathbb{Q}}$-vector spaces in question.

- Proof of part (ii): If K is a totally real subfield of $\mathbb{Q}(\zeta_N)$, then H contains the complex conjugation $\tau_c=\sigma_{-1}$. In this case we note that the statement of the proposition is trivially true as $K^-=\{0\}$, and the only odd function in $\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$ is the identically zero function. Therefore, we assume that K is totally imaginary, and let $f\in\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$ be an odd function. We write $f=\frac{1}{|H|}\sum\limits_{\sigma\in H}\sigma(f)$. Using (18), we have

$$\begin{aligned}
L(1,f)&=\frac{1}{|H|}\sum_{\sigma\in H}L(1,\sigma(f))\\
&=\frac{i\pi}{2N|H|}\sum_{\substack{n=1\\(n,N)=1}}^{N}f(n)\sum_{\sigma\in H}\sigma\left(\frac{1+\zeta_N^n}{1-\zeta_N^n}\right).
\end{aligned}$$

For every n coprime to N, we note that the inner sum is an element of K and changes sign upon the action of the complex conjugation τ_c. Hence the elements are totally imaginary and it follows that $L(1,f)\in i\pi\cdot K^-$. Therefore,

$$\left\{\ L(1,f)\ :\ f\in\mathbf{F}_{\mathbf{D}}(N)^H\text{ and }f\text{ is odd }\right\}\subseteq i\pi\cdot K^-.$$

Now we compute the dimension of the $\mathbb{Q}$-vector space on the left hand side. By Theorem 2.4, there exists no non-zero odd $f\in{\mathbf{F}_{\mathbf{D}}}^0(N)$ such that $L(1,f)=0$. Hence,

$$\begin{aligned}
&\dim_{\mathbb{Q}}\left\{L(1,f)\ :\ f\in\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H\text{ and }f\text{ is odd }\right\}\\
&=\dim_{\mathbb{Q}}\left\{f\ :\ f\in\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H\text{ and }f\text{ is odd }\right\}\\
&=\frac{\varphi(N)}{2|H|},
\end{aligned}$$

where we get the last equality by noting that $\dim_{\mathbb{Q}}\mathbf{F}_{\mathbf{D}}(N)^H=\frac{\varphi(N)}{|H|}$ and that the set of even and odd functions in this vector space is a subspace of equal dimensions. On the other hand, since K is a totally imaginary field, we have

$$\dim_{\mathbb{Q}} K^- = \frac{\dim_{\mathbb{Q}} K}{2} = \frac{\varphi(N)}{2|H|}.$$

This completes the proof. ☐

Before proceeding to prove Theorem 1.3, we establish the following technical lemma.

Lemma 5.1. *Let L be an abelian number field and $\alpha \in \mathcal{O}_L^*$. Then $\log(\alpha) = \log|\alpha| + r_\alpha\,(i\pi)$ for some rational number r_α.*

Proof. Note that L is either totally real or totally imaginary. By Dirichlet's Unit Theorem, the rank of the unit group of L is same as the rank of the unit group of $L \cap \mathbb{R}$. Therefore, for any $\alpha \in \mathcal{O}_L^*$, there exists a natural number n such that $\alpha^n \in \mathbb{R}$. Since $\log(\alpha^n) - n\log(\alpha) \in 2\pi i\,\mathbb{Z}$ for any integer n; the result of the lemma holds. ☐

5.1. **Proof of Theorem 1.3.** For a function $f \in \mathbf{F}_{\mathbf{D}}(N)$, we decompose $f = f_e + f_o$, where

$$f_e(n) := \frac{f(n)+f(-n)}{2} \qquad \text{and} \qquad f_o(n) = \frac{f(n)-f(-n)}{2}.$$

Note that $f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$ if and only if $f_e,\ f_o \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. Furthermore, the values $L(1, f_e)$ and $L(1, f_o)$ are linearly independent over the field of algebraic numbers. This fact follows from Theorem 2.5. Thus, to prove Theorem 1.3, it suffices to restrict ourselves to the case of even functions f_e and odd functions f_o.

Proof of Theorem 1.3. The first equality of spaces is established in Proposition 5.1. We now prove

$$\mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H = \left\{ f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N) \,:\, L(1,f) \in K\left\langle \{\log\alpha \mid \alpha \in \mathcal{O}_K^*\}\right\rangle \right\}$$

by decomposing $f = f_e + f_o$ and applying Proposition 5.2.

Let $f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$, so that $f_e, f_o \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. Then, for any $0 \le b \le N-1$ and $\sigma_n \in H$ for which $\sigma_t \in H$ such that $nt \equiv 1 \bmod N$,

$$\begin{aligned}
\sigma_n^{-1}\left(\widehat{f}(b)\right) &= \frac{1}{\varphi(N)} \sum_{\substack{a=1\\(a,N)=1}}^{N} f(a)\, e^{2\pi i abt/N} = \frac{1}{\varphi(N)} \sum_{\substack{m=1\\(m,N)=1}}^{N} f(mn)\, e^{2\pi i bm/N} \\
&= \frac{1}{\varphi(N)} \sum_{\substack{m=1\\(m,N)=1}}^{N} \sigma_n^{-1}(f)(m)\, e^{2\pi i bm/N} = \widehat{f}(b),
\end{aligned}$$

as $f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. Thus, $\widehat{f}(b) \in K$ for all $1 \le b \le N-1$. Therefore, from (17), we obtain that $L(1,f)$ is a K-linear combination of $\{\log(1-\zeta_N^b) \;:\; 1 \le b \le N-1\}$. Hence, $L(1, f_e)$ is a linear form in logarithms of elements in $\{\,|1-\zeta_N^b| \;:\; 1 \le b \le N/2\}$ with coefficients in K. On the other hand, by Proposition 5.2, we have

$$L(1, f_e) \in \overline{\mathbb{Q}}\left\langle \{\log(|u|) \,:\, u \in \mathcal{O}_{K^+}^*\}\right\rangle.$$

Applying Lemma 2.1, we note that $L(1, f_e) \in K\left\langle\{\log(|u|) \,:\, u \in \mathcal{O}_K^*\}\right\rangle$. Since $f_o \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$, by Proposition 5.2 we have that $L(1, f_o) \in i\pi \cdot K^-$ and $i\pi = (N/2)\log(\zeta_N)$. Therefore,

$$L(1,f) = L(1,f_e) + L(1,f_o) \in K\left\langle\{\log(\alpha) \,:\, \alpha \in \mathcal{O}_K^*\}\right\rangle.$$

Conversely, let $f \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)$ such that

$$L(1,f) \in K\left\langle\{\log(\alpha) \,:\, \alpha \in \mathcal{O}_K^*\}\right\rangle.$$

By Lemma 5.1, we can write $\log(\alpha) = \log(|\alpha|) + i\pi r_\alpha$ for some rational number r_α in the expression for $L(1, f)$. Thus, we deduce that $L(1, f_e)$ is a $\overline{\mathbb{Q}}$-linear combination of $\log|\alpha|$'s, and $L(1, f_o)$ is an algebraic multiple of $i\pi$. That is, we obtain

$$L(1, f_e) \in K\left\langle \{\log(|\alpha|) : \alpha \in \mathcal{O}_K^*\}\right\rangle \text{ and } L(1, f_o) \in i\pi \cdot K.$$

Furthermore noting that both $L(1, f_e)$ and $L(1, f_o)$ are real valued, we have

$$L(1, f_e) \in K^+\left\langle \{\log(|\alpha|) : \alpha \in \mathcal{O}_K^*\}\right\rangle \text{ and } L(1, f_o) \in i\pi \cdot K^-.$$

Hence, by Proposition 5.2, we deduce that there exists an odd function $h_o \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$ such that $L(1, f_o) = L(1, h_o)$ and there exist some even functions $g_j \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$ and $c_j \in \overline{\mathbb{Q}}$ such that

$$L(1, f_e) = \sum_{j \in I} c_j \, L(1, g_j),$$

for some finite index set I. By Theorem 2.4, we conclude that $f_o = h_o$, so that $f_o \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. Moreover, by Theorem 2.3, we have $f_e = \sum_{j \in I} c_j \, g_j$. We now show that $f_e \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. For $\sigma_n \in H$, and $(b, N) = 1$ we have

$$\sigma_n^{-1}(f_e)(b) = f_e(bn) = \sum_{j \in I} c_j \, g_j(bn) = \sum_{j \in I} c_j \, g_j(b) = f_e(b),$$

where we use that $g_j \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. This shows that $f_e \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. Therefore, $f = f_o + f_e \in \mathbf{F}_{\mathbf{D}}^{(\mathbf{0})}(N)^H$. This proves the theorem. □

Proof of Theorem 1.4. The second equality stated in theorem is immediate from Theorem 1.3 once we take $K = \mathbb{Q}(\zeta_q)$. The first equality follows from Corollary 5.1. This completes the proof. □

6. **Concluding Remarks**

For any positive integer $k \geq 1$, let

$$\psi_k(x) := \frac{d^k}{dx^k}\left(\psi(x)\right)$$

be the k-th order polygamma function. A natural extension of our study is to understand the expression of an N-division value of the polygamma function in terms of primitive N-division values. We connected the study of linear relations among division values of the digamma function to those among the value $L(1, f)$ for $f \in \mathbf{F}^{(\mathbf{0})}(N)$. Analogously, the division values of the k-order polygamma function are related to the special value

$$L(k+1, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^{k+1}}.$$

Thus, one may canonically generalize our considerations to this setting.

With the utility of the induction operator evident in our work, it would be interesting to explore such operators for other cognate functions. We relegate the development of these ideas to future research.

Data Availability Statement: This manuscript has no associated data.
Conflict of Interest Statement: The authors have no competing interests to declare.

Acknowledgments

We thank Sunil Naik, Pieter Moree and both the referees for helpful comments and suggestions on an earlier draft of this paper. The first author thanks Chennai Mathematical Institute for a pleasant working environment.

References

[1] A. Baker, *Transcendental number theory*, Cambridge University Press, Cambridge, (1975), xiv+169.

[2] A. Baker, B. J. Birch, E. A. Wirsing *On a problem of Chowla* J. Number Theory, **5**,(1973),224-236.

[3] A. Bharadwaj, S. Pathak *On special values of Dirichlet series with periodic coefficients.* J. Number Theory **238**, (2022) 978-1011

[4] T. Chatterjee, M. R. Murty, S. Pathak, *A vanishing criterion for Dirichlet series with periodic coefficients*, Contemporary Math., Vol. **701** (2018) 69-80.

[5] S. Chowla, *A special infinite series*, Kong. Norsk. Vidensk. Selsk. Forhandl. **37** (1964), 85-87. (See also, S. Chowla, Collected Papers, Volume 3, pp. 1048-1050, Centre de Recherches Mathématiques, 1999.)

[6] T. Chatterjee, S. Dhillon, *On a conjecture of Murty-Saradha about digamma values*, Monatsh. Math., **199**, (2022), 1, 23-43.

[7] K. Girstmair, *Some linear relations between values of trigonometric functions at $k\pi/n$*, Acta Arith., **81**, (1997), 4, 387-398.

[8] M. R. Murty, V. K. Murty, *A problem of Chowla revisited*, J. Number Theory, **131**, (2011), 9, 1723-1733.

[9] M. R. Murty, N. Saradha, *Transcendental values of the digamma function*, J. Number Theory, **125**, (2007), 2, 298-318.

[10] M. R. Murty and S. Pathak, *The Okada space and vanishing of $L(1, f)$*, Functiones et Approximatio Commentarii Mathematici, **66** no.1 (2022) 35-57.

[11] T. Okada, *On a certain infinite series for a periodic arithmetic function*, Acta Arithmatica, **40** (1982) 143-153.

[12] T. Okada, *Dirichlet series with periodic algebraic coefficients*, Journal of the London Math. Soc., **33** no. 2 (1986) 13-21.

[13] S. Pathak, *Special values of L-series, periodic coefficients and related themes*, PhD thesis, Queen's University (2019).

[14] H. M. Stark, *L-functions at $s = 1$. II. Artin L-functions with rational characters*, Advances in Math.,**17**,(1975),1,60-92.

[15] L. C. Washington , *Introduction to cyclotomic fields*, Graduate Texts in Mathematics, **83**, 2^{nd} edition, Springer-Verlag, New York, (1997), xiv+487.

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