Estimating the Growth Rate of the Zeta-function Using the Technique of Exponent Pairs

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Abstract

The Riemann-zeta function is of prime interest, not only in number theory, but also in many other areas of mathematics. One of the greatest challenges in the study of the zeta function is the understanding of its behaviour in the critical strip $0 < \Re(s) < 1$, where $s = \sigma + it$ is a complex number. In this note, we first obtain a crude bound for the growth of the zeta function in the critical strip directly from its approximate functional equation and then show how the estimation of its growth reduces to a problem of evaluating an exponential sum. We then apply the technique of exponent pairs to get a much improved bound on the growth rate of the zeta function in the critical strip.

1 Introduction

The Riemann zeta function $\zeta(s)$ with $s = \sigma + it$ a complex number can be defined in either of the two following ways - $\zeta(s) = \sum_n \frac{1}{n^s}$ or $\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}$, where in the first expression we sum over all natural numbers while in the second, we take the product over all primes. That these two expressions are equivalent follows directly from the expression of $n$ as a product of primes and the fundamental theorem of arithmetic. It’s easy to check that both the Dirichlet series, i.e., the first sum and the infinite product converge uniformly in any finite region with $\sigma > 1$. This follows because $|n^s| = |n^{\sigma+it}| = |n^\sigma||n^it| = |n^\sigma||e^{it\log n}| = |n^\sigma|$ since $|e^{it\log n}| = 1$ and then, of course, we know that $\sum_n n^{-\sigma}$ converges only for $\sigma > 1$.

Our goal in this note is to discuss the behaviour of $\zeta(s)$ in the critical strip $0 \leq \sigma \leq 1$. More precisely, we want to inspect how $\zeta(s)$ varies with $t$, the imaginary part of $s$, in an infinite vertical strip in this critical region. One of the most intriguing hypothesis in this direction is the Lindelof hypothesis. It states that for any $\epsilon > 0$, $\zeta(\frac{1}{2} + it) << t^\epsilon$, where $<<$ denotes order.
While the Lindelof hypothesis is unproven, it was shown by Backlund that the Lindelof hypothesis is equivalent to the following statement about the zeros of the zeta function: for every \( \epsilon > 0 \), the number of zeros with real part at least \( \frac{1}{2} + \epsilon \) and imaginary part between \( t \) and \( t + 1 \) is \( o(\log(t)) \) as \( t \) tends to infinity. The Riemann hypothesis implies that there are no zeros at all in this region and so implies the Lindelof hypothesis. The number of zeros with imaginary part between \( t \) and \( t + 1 \) is known to be \( O(\log(t)) \), so the Lindelof hypothesis seems only slightly stronger than what has already been proved, but in spite of this it has resisted all attempts to prove it and is very hard.

For any real \( \sigma \), we define by \( \mu(\sigma) \) the infimum of the numbers \( \delta \) such that \( \zeta(\sigma + it) < t^\delta \). What we’re going to discuss is how to obtain a good bound on \( \mu(\sigma) \) for \( 0 < \sigma < 1 \). By the technique of exponent pairs, we’ll estimate \( \mu(\frac{1}{2}) \), obtaining a constant smaller than \( \frac{1}{6} \). The Lindelof hypothesis, of course, claims that this constant is arbitrarily small, but the best possible bound available today for \( \mu(\frac{1}{2}) \) is approximately 0.155. While we won’t be able to obtain that bound by the process we discuss in this note itself, the bound we obtain by exponent pairs will only be marginally worse than that.

The zeta function is known to satisfy the following functional equation.

\[
\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{1}{2})}
\]

Writing \( \chi(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \), we get the usual standard form of the functional equation

\[
\zeta(s) = \chi(s) \zeta(1-s).
\]

It’s easy to verify from the properties of the Gamma function that

\[
\chi(s)\chi(1-s) = 1.
\]

It also follows from the general theory of Dirichlet series that, as a function of \( \sigma \), \( \mu(\sigma) \) is continuous, non-increasing and convex downwards. Since \( \zeta(s) \) is bounded for \( \sigma > 1 \), it follows that \( \mu(\sigma) = 0 \) for \( \sigma > 1 \) and so, from the functional equation, that \( \mu(\sigma) = \frac{1}{2} - \sigma \) for \( \sigma < 0 \). By continuity, these equations continue to hold for \( \sigma = 1 \) and \( \sigma = 0 \) respectively. The line joining the points \( (0, \frac{1}{2}) \) and \( (1,0) \) on the graph of \( y = \mu(\sigma) \) has the equation \( y = \frac{1}{2} - \frac{1}{2} \sigma \). So, on invoking convexity, we now obtain that \( \mu(\sigma) \leq \frac{1}{2} - \frac{1}{2} \sigma \) for \( 0 < \sigma < 1 \). In particular, for \( \sigma = \frac{1}{2} \), we get that \( \mu(\frac{1}{2}) \leq \frac{1}{4} \), that is, \( \zeta(\frac{1}{2} + it) < t^{\frac{1}{4} + \epsilon} \) \( \forall \epsilon > 0 \).

A slight improvement in this bound may be made by considering the following theorem, often called the approximate functional equation of the zeta function.
We state this theorem here without proof. However, application of this theorem will only remove the small constant $\epsilon$ in the exponent. To get better bounds, we somehow have to ensure some sort of cancelling between the terms of the sum involved in the zeta function. That is precisely where the theory of exponent pairs help us.

**Theorem 1. (Approximate functional equation)** Let $h$ be a positive constant and $x, y$ be two real numbers such that $x > h > 0$ and $y > h > 0$. Suppose that $s = \sigma + it$ with $0 < \sigma < 1$ and $t = 2\pi xy$. Then

$$
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^s} + O(x^{-\sigma} \log |t|) + O(|t|^{\frac{1}{2} - \sigma} y^{\sigma - 1}).
$$

Putting $\sigma = \frac{1}{2}$ and $x = y = \sqrt{\frac{t}{2\pi}}$, we get $\zeta(\frac{1}{2} + it) \ll O(t^{\frac{1}{4}})$.

## 2 Exponential sums and Preliminary Lemmas

An exponential sum is any sum of the form $S = \sum_{n \in I} e(f(n))$ where $f$ is a real valued function, $I$ is a real closed interval with integer endpoints and $e(x)$ denotes $e^{2\pi ix}$. We may occasionally need to assume various 'nice' properties of the function $f$ and, in general, the more information we have about $f$, the better our estimate of such a sum will be. First of all, we note that by the triangle inequality, we have that $|S| \leq |\sum_{n \in I} e(f(n))| \leq |I|$ where $|I|$ is the length of our interval since $|e(f(n))|$ has absolute value 1. This estimate is the trivial estimate and it’s easy to see that it actually holds with equality when $f(n) = xn + y$ and $x$ is an integer. That means that if we want to really improve upon this trivial estimate, imposing some additional conditions on $f$ is a must.

An important trick introduced by Weil in evaluating such sums is Weil differencing. Note that if $S$ is the sum above, then

$$
|S|^2 = \sum_{a \leq m, n \leq b} e(f(m) - f(n)) = \sum_{|h| \leq b-a} \sum_{n \in I_h} e(f(n + h) - f(n))
$$

where $I_h = \{n | a \leq n, n + h \leq b\}$. This technique is helpful because the difference function $f(n + h) - f(n)$ is usually much easier to handle than the original function $f$. For example, if the function $f$ is a polynomial, then this trick allows us to reduce the degree of the polynomial by one and, by repeatedly applying this trick, we can get a degree one polynomial in the exponent and this makes the problem of estimation trivial because exponential sums of linear functions are nothing but geometric series.

However, Weil’s method was considerably improved by Van der Corput by a clever application of the Cauchy-Schwarz inequality. It’s Van der Corput’s inequality which we shall use to derive the two processes of generating new exponent pairs from old ones and so we begin by proving the so-called Van der Corput’s inequality.

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From now on, for convenience, we’ll take our interval \( I \subseteq (N, 2N] \) for some integer \( N \) and our estimates will involve \( N \) instead of \( |I| \).

**Theorem 2.** *(Weil-Van der Corput inequality)* Suppose that \( g(n) \) is a complex-valued function which is zero outside our given interval \( I \). Then for any positive integer \( H \),

\[
|\sum_n g(n)|^2 \leq \frac{|I| + H}{H} \sum_{|h| < H} (1 - \frac{|h|}{H}) \sum_n g(n)g(n-h).
\]

In particular, if \( g(n) = e(f(n)) \) if \( n \in I \) and zero otherwise, and if \( S = \sum_{n \in I} e(f(n)) \), then

\[
|S|^2 \leq \frac{|I| + H}{H} \sum_{|h| < H} |S_1(h)|
\]

with \( S_1(h) = \sum_{n \in I(h)} e(f(n+h) - f(n)) \) and \( I(h) = \{n|n \in I, n+h \in I\} \).

**Proof.** We observe that \( H \sum_n g(n) = \sum_{k=1}^H \sum_n g(n+k) = \sum_n \sum_{k=1}^H g(n+k) \). The inner sum is zero unless \( a-H < n \leq b-1 \). So, applying the Cauchy-Schwarz inequality, we get,

\[
H^2 |\sum_n g(n)|^2 \leq (|I|+H) \sum_n |\sum_{k=1}^H g(n+k)|^2 = (|I|+H) \sum_{l=1}^H \sum_{k=1}^H \sum_n g(n+k)g(n+l).
\]

Collecting together all terms with \( l-k = h \), we get that

\[
|\sum_n g(n)|^2 \leq \frac{|I| + H}{H} \sum_{|h| < H} (1 - \frac{|h|}{H}) \sum_n g(n)g(n-h).
\]

Now, we define \( S_1(h) = \sum_{n \in I(h)} e(f(n+h) - f(n)) \) with \( I(h) = \{n|n \in I, n+h \in I\} \).

Then it follows that \( |S|^2 \leq \frac{|I| + H}{H} \sum_{|h| < H} |S_1(h)| \).

\[
\square
\]

Since the theorem above holds for any positive integer \( H \), we take \( H \leq |I| \) to obtain

\[
|S|^2 \leq \frac{2|I|}{H} \sum_{|h| < H} |S_1(h)|.
\]

If \( H \) is not an integer but an arbitrary positive number, we let \( H_0 = \lceil H \rceil + 1 \).
and get that

\[ |S|^2 \leq \frac{2|I|}{H} \sum_{|h| < H} |S(h)| \leq \frac{2|I|}{H} \sum_{|h| < H} |S_1(h)|. \]

Further, if \( h > 0 \), then

\[ S_1(h) = \sum_{a < n \leq b - h} e(f(n + h) - f(n)) \]

and

\[ S_1(-h) = \sum_{a + h < n \leq b} e(f(n + h) - f(n)). \]

By making a change of variables, we see that

\[ S_1(-h) = S_1(h) \]

So ultimately we obtain that

\[ |S|^2 \leq \frac{2|I|^2}{H} + \frac{4|I|}{H} \sum_{1 \leq h \leq H} |S_1(h)|. \]

Since we have assumed that \( I \subseteq (N, 2N] \) for some integer \( N \), we may also write that

\[ |S|^2 \ll \frac{N^2}{H} + \frac{N}{H} \sum_{1 \leq h \leq H} |S_1(h)|. \]

The Weil-van der Corput inequality is the basic ingredient of the so-called A-process we shall now discuss. Both the A and B processes allow us to sharpen estimates about a given exponential sum from some already proven estimate. While the A-process depends upon the above inequality, the B-process involves an approximation formula for evaluation of exponential integrals, usually referred to as the method of stationary phase. We state this formula here without proof.

**Theorem 3.** (Method of Stationary Phase) Suppose that the function \( f \) has four continuous derivatives in \( I \) and \( f'' < 0 \) in \( I \). With \( I \subseteq (N, 2N] \), let \( a, b \) be the endpoints of \( I \) and let \( \alpha = f'(b), \beta = f'(a) \). Assume that for some constant \( F > 0 \), \( f^2(x) \approx FN^{-2}, f^3(x) \approx FN^{-3}, f^4(x) \approx FN^{-4} \forall x \in I \). Let \( x_\nu \) be such that \( f'(x_\nu) = \nu \) and let \( \phi(\nu) = -f(x_\nu) + \nu x_\nu \). Then

\[ \sum_{n \in I} e(f(n)) = \sum_{\alpha \leq \nu \leq \beta} \frac{e(-\phi(\nu) - \frac{1}{8})}{|f^m(x_\nu)|^{\frac{1}{2}}} + \text{error terms.} \]

The error terms in the above theorem can be shown to be \( << FN^{-1} + F^{-\frac{1}{2}} N \), but since we won’t be going into any rigorous proof of the estimate, we don’t look into it.

### 3 The A and B Processes for Generating New Exponent Pairs

We shall deal with only those exponential sums \( \sum_{n} e(f(n)) \) for which the function \( f \) satisfies certain conditions. For a rigorous treatment, the exact conditions may be complicated, but the most important condition we shall need is that \( f'(x) \approx yx^{-s} \) for some \( y > 0 \) and some \( s > 0 \). We define \( L = yN^{-s} \) so that \( f \approx L \). Then if \( S = \sum_{n \in I} e(f(n)) \ll L^k N^l \), we say that \((k, l)\) is an exponent.
pair. By looking at the trivial estimate, we immediately notice that \((0,1)\) is an exponent pair. The \(A\) and \(B\) processes are basically two methods to generate a new exponent pair from a known one. By invoking a sequence of these processes, we may obtain various exponent pairs and thereby obtain a good estimate of our exponential sum. Though algorithms exist for determining which sequence of these processes yields the optimal pair, we won’t go into that. For obtaining a suitably good estimate in case of, say, the growth of the zeta function, it’s enough to look up a table of exponent pairs and see which of them gives the best possible estimate. We now state the \(A\) and \(B\) processes. We don’t give a rig

**Theorem 4.** (A-process) If \((k,l)\) is an exponent pair, then so is \((\frac{k}{2k+2}, \frac{k+l+1}{2k+2})\).

**Proof.** We denote \(\sum_{n \in I} e(f(n))\) by \(S\) throughout. By the Weil-van der Corput inequality, we see that if \(H \leq N\), then

\[ |S|^2 \ll \frac{N^2}{H} + \frac{N}{H} \sum_{1 \leq h \leq H} |S_1(h)|, \]

where \(S_1(h) = \sum_{a < n < b, a < n + h \leq b} e(f_1(n, h))\) with \(f_1(n, h) = f(n + h) - f(n)\). If \(f' \approx L\), then heuristically, \(f_1' \approx L|hN^{-1}\). If we assume that the exponent pair \((k, l)\) can be applied to \(S_1(h)\), then we obtain that

\[ |S|^2 \ll \frac{N^2}{H} + \frac{N}{H} \sum_{1 \leq h \leq H} (L|hN^{-1})kN^l \ll N^2H^{-1} + H^kL^kN^{1-k+l}. \]

The minimum is attained when the two terms in the above sum are equal. Equating the two terms, we get, on simplification, \(S \ll L^{\frac{1}{2}}N^\frac{1}{2}N^k\). So, we see that \((\frac{k}{2k+2}, \frac{k+l+1}{2k+2})\) is indeed an exponent pair.

**Theorem 5.** (B-process) If \((k,l)\) is an exponent pair, then so is \((l - \frac{1}{2}, k + \frac{1}{2})\).

**Proof.** By theorem 3, we have that \(S = e(-\frac{1}{2}) \sum_{a \leq \nu \leq \beta} e(-\phi(\nu)) |f''(x_\nu)|^\frac{1}{2} + \text{error terms.}\) Ignoring the error terms as part of our heuristic approach for the time being, we see that \(\phi'(\nu) = x_\nu \approx N\) and \(f''(x_\nu) \approx LN^{-1}\). So, by partial summation, we get that

\[ \sum_{a \leq \nu \leq \beta} e(-\phi(\nu)) |f''(x_\nu)|^\frac{1}{2} \ll L^{-\frac{1}{2}}N^\frac{1}{2} \min_{\beta' \leq \beta} |\sum_{a < \nu \leq \beta} e(\phi(\nu))|. \]

Assuming that the exponent pair \((k, l)\) can be applied to the last sum, we get \(S \ll L^{-\frac{1}{2}}N^\frac{1}{2}N^kL^l \ll L^{1-\frac{k}{2}}N^{k+\frac{1}{2}}.\) So, we conclude that \((l - \frac{1}{2}, k + \frac{1}{2})\) is indeed an exponent pair.
We shall now apply exponent pairs generated by the $A$ and $B$ processes to the zeta function in the critical strip. First, however, we prove a couple of lemmas which reduces the problem of bounding the growth of the zeta function to that of the evaluation of an exponential sum.

4 The Zeta Function as an Exponential Sum

Application of Exponent Pairs

Before we can apply the theory of exponent pairs to the zeta function, we need a few results that will enable us to reduce the expression for the zeta function to an exponential sum. We prove the Kusmin-Landau inequality first and then using it, reduce the sum involved in the expression of the zeta function to an exponential sum.

**Theorem 6.** *(Kusmin-Landau estimate)* If, in the interval $I$, $f$ is continuously differentiable, $f'$ is monotonic and $||f'|| \geq \lambda > 0$ where $||.||$ denotes the distance from the nearest integer, then $\sum_{n \in I} e(f(n)) << \lambda^{-1}$.

**Proof.** Since $|\sum_{n \in I} e(f(n))| = |\sum_{n \in I} e(-f(n))|$, we may assume that $f'$ is increasing. Our hypothesis implies that for some integer $k$, $k + \lambda \leq f'(n) \leq k + 1 - \lambda$. But since $\sum_{n \in I} e(f(n)) = \sum_{n \in I} e(f(n) - kn)$, we may assume that $\lambda \leq f'(n) \leq 1 - \lambda$.

Define $g(n) = f(n+1) - f(n)$. By mean value theorem, we see that for some $x_n$ with $n \leq x_n \leq n + 1$, $g(n) = f'(x_n)$. So, $g$ is increasing and $\lambda \leq g(n) \leq 1 - \lambda$.

We write $e(f(n)) = \frac{e(f(n)) - e(f(n+1))}{1 - e(g(n))} = \left( e(f(n)) - e(f(n+1)) \right) C_n$ where $C_n = \frac{1}{2} (1 + i \cot \pi g(n))$.

Thus, $\sum_{n \in I} e(f(n)) = \sum_{n=a+1}^{b-1} \left\{ e(f(n)) - e(f(n+1)) \right\} C_n + e(f(b))$

$= \sum_{n=a+2}^{b-1} e(f(n))(C_n - C_{n-1}) + e(f(a+1))C_{a+1} + e(f(b))(1 - C_{b-1})$

and thus $|\sum_{n \in I} e(f(n))| \leq \frac{1}{2} \sum_{n=a+2}^{b-1} |\cot(\pi g(n-1)) - \cot(\pi g(n))| + |C_{a+1}| + |1 - C_{b-1}|$.

Since $\cot(\pi g(n))$ is a decreasing function, we remove the absolute value sign from the right hand sum. Writing the remaining sum as a telescopic series, we get $|\sum_{n \in I} e(f(n))| \leq \frac{1}{2} \left\{ \cot(\pi g(a+1)) - \cot(\pi g(b - 1)) \right\} + |C_{a+1}| + |1 - C_{b-1}|$.

We now get the desired result by using the bound $|\cot \pi x| << ||x||^{-1}$. □
Theorem 7. Suppose that $\frac{1}{2} \leq \sigma < 1$ and $N \leq M$. Then we have that

$$\sum_{N < n \leq M} n^{-\sigma-it} \ll N^{-\sigma} \max_{N < u \leq M} \left| \sum_{N < n \leq u} n^{-it} \right|.$$ 

Proof. Let $S(u) = \sum_{N < n \leq u} n^{-it}$. Then, by partial summation,

$$\sum_{N < n \leq M} n^{-\sigma-it} = S(M)M^{-\sigma} + \sigma \int_M^\infty S(u)u^{-\sigma-1}du \ll N^{-\sigma} \max_{N < u \leq M} |S(u)|$$

and we’re done.

Theorem 8. Suppose that $\frac{1}{2} \leq \sigma < 1$ and $t \geq 3$. Then we have that

$$|\zeta(\sigma + it)| \ll \left| \sum_{n \leq t} n^{-\sigma-it} \right| + t^{1-2\sigma} \log t.$$ 

Proof. If $\sigma > 1$ and $M \geq 1$, then

$$\zeta(s) = \sum_{n \leq M} n^{-s} + \int_M^\infty u^{-s}d[u] = \sum_{n \leq M} n^{-s} + \frac{M^{1-s}}{s-1} + s \int_M^\infty \frac{|u-u|}{u^{s+1}}du.$$ 

The last integral converges for $\sigma > 0$ and so gives an analytic continuation of $\zeta(s)$ to the region $\sigma > 0, s \neq 1$. Putting $M = t^2$ and using the bound $|u-u| \leq 1$, we get that $\zeta(s) = \sum_{n \leq t^2} n^{-s} + O(t^{1-2\sigma}).$

The sum over $t \leq n \leq t^2$ can be divided into $\ll \log t$ sums of form $\sum_{N < n \leq N_1} n^{-\sigma-it}$ with $N_1 = \min(2N, t^2)$. Using Theorems 6 and 7, we see that each such subsum is $\ll N^{1-\sigma}t^{-1} \ll t^{1-2\sigma}$, and the result follows.

Now, we come to our most important theorem.

Theorem 9. Let $t \geq 3$ and assume that $(k, l)$ is an exponent pair with $k + 2l \geq \frac{3}{2}$. Let $\theta(k, l) = \frac{(2k+2l+1)}{4}$. Then $\zeta(\frac{1}{2} + it) \ll t^{\theta(k, l)} \log t$.

Proof. By looking at Theorems 7 and 8, we see that it suffices to show that

$$N^{-\frac{1}{2}} \sum_{N < n \leq N_1} n^{-it} \ll t^{\theta(k, l)} \text{ with } 1 \leq N \leq t \text{ and } N_1 \leq 2N.$$ 

We note that $n^{-it} = e^{2\pi i (n \log n)}$ and so, in this case, $f(n) = \frac{1}{2\pi} \log n$. So, $L \approx f'(n) \approx \frac{1}{N}$. Now, we use the exponent pairs $(k, l)$ and $B(k, l) = (l-\frac{1}{2}, k+\frac{1}{2})$.

We get that $N^{-\frac{1}{2}} \sum_{N < n \leq N_1} n^{-it} \ll \min(t^k N^{l-k-\frac{1}{2}}, t^{l-\frac{1}{2}} N^{k-l+\frac{1}{2}})$.

The result now follows from the fact that for any real numbers $a, b$, $\min(a, b) \leq \sqrt{ab}$ and noting that there are $\ll \log t$ diadic intervals of form $(N, N_1]$ with $N_1 \leq 2N$.
Using our Theorem 9, we may estimate similarly \( \zeta(\sigma + it) \) for any \( 0 < \sigma < 1 \). In that case, we get, as in the proof of Theorem 9, \( \zeta(\sigma + it) << t^{\theta(k,l)} \log t \) with \( \theta(k,l) = \frac{k+2l-\sigma}{2} \) whenever \( k + 2l > 1 + \sigma \). By this estimate, we can bound the growth of the zeta function in any vertical line in the critical strip. In particular, for \( \sigma = \frac{1}{2} \), we may use \( (k,l) = ABABAB(0,1) = (\frac{11}{82}, \frac{57}{82}) \) and see that with this, the estimate for \( \theta(k,l) = \frac{27}{164} \approx 0.165 \).

So we obtain \( \zeta(\frac{1}{2} + it) << t^{\frac{27}{164}} \log t << t^{\frac{27}{164} + \epsilon} \). We may do this for other values of \( \sigma \) as well. Note that the exponent pair \( (\frac{11}{82}, \frac{57}{82}) \) is not necessarily the optimal one. Other pairs may give a slightly better bound and indeed, there is an algorithm to determine the optimal exponent pair. The algorithm provides us with the optimal sequence of \( A, B \) operations. Since the \( B \) process has order 2, i.e., repeating it twice yields the original exponent pair, any such optimal sequence is of the form \( A^i B A^{i'} B \ldots \). We will not go on to describe the algorithm.

5 Concluding Remarks

The Lindelof hypothesis, as we already mentioned, says that \( \mu(\frac{1}{2}) < \epsilon \forall \epsilon > 0 \). Using the technique of exponent pairs, we have thus been able to achieve a bound of about 0.165 on \( \mu(\frac{1}{2}) \) and have seen how to obtain a reasonable estimate of \( \mu(\sigma) \) for any \( \sigma \in (0,1) \). The best possible bound known today for \( \mu(\frac{1}{2}) \) is around 0.155, but that uses other techniques. A result of Bombieri-Iwaniec gives a bound of about 0.161 on \( \mu(\frac{1}{2}) \), and that result involves a modification of Van der Corput’s method and uses a Fourier series approximation of the integral involved in the \( B \) process to lower the bound marginally. However, proving \( \mu(\frac{1}{2}) \) to be arbitrarily small, as suggested by the Lindelof hypothesis, remains very difficult.

References


