CMI GRADUATE COURSE NOTES (LAZARSFELD I)

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By a variety, we will mean (unless otherwise stated) a complex quasi-projective variety, usually reduced and irreducible, and for the most part normal. We will work in the Zariski topology and with algebraic functions. Remember that Zariski open sets are very big – for example, on a compact Riemann surface (=smooth projective curve) such an open set is the complement of finitely many points. Any closed set is a finite union of irreducible closed sets. Given an irreducible closed subset Y, the subset Y^{reg} of its smooth points is open and dense in Y. If Y has codimension ≥ 2, any regular function on $X \setminus Y$ extends across Y – this is an important consequence of the normality of X; another is that $X \setminus X^{reg}$ is of codimension ≥ 2. Given an irreducible codimension one closed subset D, $D^{reg} \cap X^{reg}$ is dense in D. (We can safely take the codimension of an irreducible closed set Y to mean $dim\ X - dim\ Y$.)

For the reader's convenience, whenever possible I adopt the notations from Lazarsfeld's book.

1. Line Bundles, Cartier/Weil Divisors, Linear Systems

1.1. Line Bundles and Invertible Sheaves. Let X be a variety. Let r be a positive integer. By a locally free sheaf $\underline{E}$ on X of rank r, we mean a coherent (i.e., finitely generated) sheaf of $\mathcal{O}_X$-modules which is locally free. In other words, we are given a sheaf of complex vector spaces

$$U \mapsto \underline{E}(U)$$

where

- for each open set U, the complex vector space $\underline{E}(U)$ is also a finitely generated module over the ring of functions $\mathcal{O}_X(U)$,
- for $U' \subset U$, the restriction maps $\underline{E}(U) \to \underline{E}(U')$ are morphisms of $\mathcal{O}_X(U)$-modules,
- and every point of X has a neighbourhood U such that $\underline{E}(U)$ is freely generated by r elements of $\underline{E}(U)$, which (after restriction) also freely generate $\underline{E}(U')$ for every open $U' \subset U$. (A more elegant way to say this is that the r sections on U give a sheaf isomorphism $\mathcal{O}_U^r \to \underline{E}|_U$. We denote by $\mathcal{O}_U^r$ the direct sum of r copies of $\mathcal{O}_U$.)

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These notes are supplied on an as-is-where-is basis and will be periodically corrected and updated without warning.

By a vector bundle on X of rank r, we mean a variety E together with some additional structures. What follows may not be a minimal characterisation, but we should keep the complete package in mind. This consists of

(1) a surjective morphism $\pi : E \to X$,
(2) a distinguished section $X \to E$ ("the zero section")
(3) a morphism (of varieties) $\mathbb{C} \times E \to E \quad (\lambda, v) \mapsto \lambda v$ which preserves the fibres of π (i.e., $\pi(\lambda v) = \pi(v)$),
(4) a morphism $+ : E \times_X E \to E$ (with $\times_X$ denoting the fibre product of varieties) which preserves the fibres of π (i.e., $\pi(v + v') = \pi(v) = \pi(v')$),

with the further proviso of local triviality and "freeness", i.e., such that every point of X has a neighbourhood U such that $E|_U \equiv \pi^{-1}(U)$ is isomorphic to the product variety $U \times \mathbb{C}^r$, in a way compatible with all the structures in the above list. (For this last compatibility to make sense, one has to check that the structures given on E localise to every open set U: we leave this as an exercise.)

Given a vector bundle E, we associate to it the sheaf of modules:

$$U \mapsto \underline{E}(U) = \text{sections of } E|_U$$

and one can check that $\underline{E}$ is indeed coherent, locally free, and of rank r. On the other hand, given a coherent sheaf $\underline{E}$, one can ask if there is a variety E that "represents its sections" as above. This is certainly true if $\underline{E}$ is locally free.

We outline the construction in the case that is most important to us – when the rank $r = 1$. In this case we will say that $\underline{E}$ is "invertible" and E is a "line bundle". We will signal our shift in emphasis by using the notation $\underline{L}$ and L respectively. (By considering line bundles we avoid having to deal with matrix-valued functions and having to keep track of order of multiplication.) First a preliminary

Remark: We say a line bundle L is trivial on an open set U if it is isomorphic to the trivial line bundle $U \times \mathbb{C}$. A "trivialisation" is an actual isomorphism $\Phi_U : U \times \mathbb{C} \to L|_U$ commuting with the projection to U and respecting the line bundle structures. Giving a trivialisation is equivalent to giving a (regular) section σ_0 of $L|_U$ which is nowhere-vanishing. The morphism Φ_U and the section σ_0 determine one another via:

$$\Phi(x, z) = \sigma_0(x), \ x \in U$$

Suppose then that $\underline{L}$ is an invertible sheaf. Let us first deal with the case when $\underline{L}$ is *globally* free. That is, there is an element $\sigma_X \in \underline{L}(X)$ such that the maps

$$\mathcal{O}_X(U) \to \underline{L}(U), \quad f \mapsto f\sigma_X|_U$$

are isomorphisms. Consider the trivial line bundle $X\times\mathbb{C}$. The corresponding sheaf of sections is clearly $\mathcal{O}_X$, and the sheaf map

$$\mathcal{O}_X(U)\to\underline{L}(U),\quad f\mapsto f\sigma_X|_U$$

is an isomorphism. If $\underline{L}$ is only locally free, there exists an open cover U_α of X and $\sigma_\alpha\in\underline{L}(U_\alpha)$ such that for each i the map

$$\mathcal{O}_{U_\alpha}\to\underline{L}|_{U_\alpha}\quad f\mapsto f\sigma_\alpha$$

is an isomorphism of $\mathcal{O}_{U_\alpha}$-modules. On the overlaps $U_\alpha\cap U_\beta$ we have

$$\sigma_\alpha=\chi_{\alpha,\beta}\sigma_\beta$$

where $\chi_{\alpha,\beta}$ is a function, regular and nowhere-vanishing on $U_\alpha\cap U_\beta$. The collection of functions $\chi_{\alpha,\beta}$ satisfy

$$\chi_{\alpha,\beta}\chi_{\beta,\alpha}=1$$
$$\chi_{\alpha,\beta}\chi_{\beta,\gamma}\chi_{\gamma,\alpha}=1$$

(In other words, w.r.to the cover $\{U_\alpha\}$ we have a 1-cocycle with values in the sheaf $\mathcal{O}^*_X$.)

We now seek a line bundle $\tilde{L}$, trivialised by nowhere-vanishing sections $\tilde{\sigma}_\alpha$ on U_α, such that $\underline{\tilde{L}}$ is isomorphic to $\underline{L}$. One checks that this will be the case if on overlaps $U_\alpha\cap U_\beta$,

$$\tilde{\sigma}_\alpha=\chi_{\alpha,\beta}\tilde{\sigma}_\beta$$

for, in this case,

This tells us how the variety $\tilde{L}$ is to be constructed. Take the disjoint union

$$\sqcup_\alpha U_\alpha\times\mathbb{C}$$

and make the identifications

$$(x,z)\in(U_\alpha\cap U_\beta)\times\mathbb{C}\subset U_\alpha\times\mathbb{C}=(x,\chi_{\alpha,\beta}(x)z)\in(U_\alpha\cap U_\beta)\times\mathbb{C}\subset U_\beta\times\mathbb{C}$$

This gives us an "abstract variety" and the rest of the vector bundle package can also be implemented. What is not clear is that this is in fact quasiprojective.

Before we go on, we note two important binary operations we can perform on vector bundles. Given two bundles E and E', we can define their *direct sum* $E\oplus E'$ and *tensor product* $E\otimes E'$. As a variety, the direct sum can be identified with the fibre product $E\otimes_X E'$; in terms of the sheaf of sections,

$$\underline{E\oplus E'}=\underline{E\times_X E'}=\underline{E}\oplus\underline{E'}$$

and

$$\underline{E\otimes E'}=\underline{E}\otimes_{\mathcal{O}_X}\underline{E'}$$

<u>Remark:</u> The tensor product of two coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ is defined as follows. The assignation

$$U\mapsto\mathcal{F}_1(U)\otimes_{\mathcal{O}_X}\mathcal{F}_2(U)$$

defines a presheaf. One has to sheafify this to get $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$. This corrects a statement I made in the lecture.

We can also define the *dual* $\check{E}$ of a vector bundle; we have

$$\underline{\check{E}} = Hom_{\mathcal{O}_X}(\underline{E}, \mathcal{O}_X)$$

Under direct sums ranks add, and under tensor products they multiply. In particular, *the tensor product of two line bundles is a line bundle.* The natural map

$$\underline{L} \otimes_{\mathcal{O}_X} \underline{\check{L}} \to \mathcal{O}_X$$

is an isomorphism, and it is easy to check that *the set of isomorphism classes of line bundles on X is an abelian group.* This is why a line bundle is called an "invertible sheaf". *From now on we will set $L^{-1} = \check{L}$.*

1.2. **Weil and Cartier Divisors.** We will suppose that X is a (reduced, irreducible and) normal variety. By a *Weil divisor* on X we mean a formal sum $D = \sum_k a_k D_k$ where each D_k is an reducible irreducible subvariety of codimension 1 and $a_k \in \mathbb{Z}$. Thus a Weil divisor is an element of the free abelian group generated by the set of irreducible codimension 1 subvarieties of X.

By a *meromorphic function* on X we mean an element of the function field $\mathbb{C}(X)$. (We will denote the algebra of regular functions on X by $\mathbb{C}[X]$; if X is projective, $\mathbb{C}[X] = \mathbb{C}$.) Concretely, this is a function ϕ defined outside some closed set Y' and regular on $X \setminus Y'$ (and we identify two such if they agree on a common open subset.) We can take Y' to be minimal, in which case (by normality of X) it is a finite union of irreducible codimension 1 subvarieties D'_j and ϕ has a pole of order $a'_j > 0$ along D'_j. Suppose ϕ is not identically zero, and that the zero set $Y \subset X \setminus Y'$ is the union of irreducible subvarieties D^0_i (necessarily of codimension 1 by Krull's Hauptidealsatz), and the order of vanishing of ϕ along D^0_i is $a_i > 0$. Let D_i be the Zariski closure of D^0_i in X. We set

$$(\phi) = \sum_i a_i D_i - \sum_j a'_j D'_j$$

Thus we have associated to each nonzero meromorphic function a Weil divisor.

Note that given two nonzero meromorphic functions ϕ, ϕ', we have $(\phi \times \phi') = (\phi) + (\phi')$, and $(\phi) = (\phi')$ iff the two differ by a nowhere vanishing regular function. So we have an injective homomorphism

$$\mathbb{C}(X)^* / \{units\ in\ \mathbb{C}[X]\} \hookrightarrow \{Weil\ \ divisors\}$$

The image of this map is (by definition) the subgroup of *principal divisors.*

Suppose now that L is a line bundle on X. By a *meromorphic section* of L, we mean a section that is regular outside some closed Y'. Two such sections are regarded as equal if they agree on a nonempty open subset. We will denote by $Mer(L)$ the space of such sections. This is a one-dimensional vector space over $\mathbb{C}(X)$.

Remark: As we remarked earlier, a "trivialisation" of a line bundle L on an open U is equivalent to giving a (regular) section σ_0 of $L|_U$ which is nowhere-vanishing. Since we are dealing with irreducible varieties and working in the Zariski topology, we can regard σ_0 as a meromorphic section of L. (In particular there exist nonzero meromorphic sections!) Conversely, a nonzero meromorphic section of L defines a trivialisation of L away from its poles and zeroes.

As before, given a nonzero meromorphic section σ of L, we associate to it a Weil divisor:

$$(\sigma) = \sum_i a_i D_i - \sum_j a'_j D'_j$$

(What does it mean to say that σ has a zero or pole along a codimension one irreducible variety D? At any smooth point of $D \cap X^{reg}$ – and such points are Zariski dense in D – choose a local generator of $\underline{L}(U)$, that is, a section σ_0 that is regular and nowhere-vanishing in a neighbourhood $U \subset X$. Then $\sigma = f\sigma_0$ for some meromorphic function f. The function f has a zero or pole along D, and the order of this zero or pole is independent of all choices.)

Suppose given nonzero meromorphic sections σ, σ' (respectively) of two line bundles L, L'. then $\sigma \otimes \sigma'$ is a meromorphic section of $L \otimes L'$. A meromorphic ϕ function is nothing but a section of the trivial bundle, so one can also multiply ϕ and σ. (Of course one can see this directly.) This was implicit when we said that $Mer(L)$ is a one-dimensional vector space over the function field $\mathbb{C}(X)$. Clearly,

$$(\sigma \otimes \sigma') = (\sigma) + (\sigma')$$
$$(\phi\sigma') = (\phi) + (\sigma')$$

Given a meromorphic section σ of L, we define σ^{-1} to be the meromorphic section of L^{-1} such that $\sigma^{-1}[\sigma] = 1$. Here the brackets [ ] mean "evaluate σ^{-1} on σ". When do σ and σ' define the same Weil divisor? When $\sigma^{-1}[\sigma']$ is a unit in $\mathbb{C}[X]$. In other words, there is an isomorphism $L \to L'$ taking σ to σ'.

Summarising, we have:

Proposition 1.1. *The map $(L, \sigma) \mapsto (\sigma)$ induces an injective homomorphism of abelian groups*

$$\{\textit{pairs } (L, \sigma) \textit{ with } L \textit{ a line bundle and } \sigma \textit{ a nonzero meromorphic section}\}/\sim \;\to \{\textit{Weil divisors}\}$$

where $\sim$ signals that we take isomorphism classes.

We define a *Cartier divisor* to be the divisor associated to a nonzero meromorphic section of a line bundle. This is not the conventional definition because it works only on normal varieties. We will shortly give another, equivalent description, which generalises to the non-normal case. But first we ask: how do we characterise Cartier divisors among Weil divisors? Let σ be a meromorphic section of a line bundle L and consider the Cartier divisor (σ). Suppose given an open set U on which L is trivial. Then there exists a meromorphic section σ_0 which is regular on U and nowhere vanishing there. Then $\phi = \sigma\sigma_0^{-1}$ is a meromorphic function such that $(\phi|_U) = (\sigma|_U)$ as Weil divisors on U. *Thus every Cartier divisor is locally principal.*

Conversely, suppose given a Weil divisor $\sum_i a_k D_k$ which is locally principal. That is, suppose we can cover X by open sets U_α and find meromorphic functions ϕ_α satisfying

$$(\phi_\alpha|_{U_\alpha}) = \sum_i a_i\{D_i \cap U_\alpha\}$$

as Weil divisors on U_α. Then for each pair of indices α, β, we have a meromorphic function

$$\chi_{\alpha,\beta} = \phi_\alpha^{-1}\phi_\beta$$

which is regular and invertible on $U_\alpha \cap U_\beta$. Further, on $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$\chi_{\alpha,\beta}\chi_{\beta,\gamma}\chi_{\gamma,\alpha} = 1$$

So there exists a line bundle L with meromorphic sections σ_α such that σ_α is regular and nowhere-vanishing on U_α and such that on $U_\alpha \cap U_\beta$ we have

$$\sigma_\alpha = \chi_{\alpha,\beta}\sigma_\beta$$

For each α define the meromorphic section τ_α by

$$\tau_\alpha = \phi_\alpha\sigma_\alpha$$

On $U_\alpha \cap U_\beta$ we have

$$\tau_\alpha = \phi_\alpha\sigma_\alpha = \phi_\alpha\chi_{\alpha,\beta}\sigma_\beta = \phi_\beta\sigma_\beta = \tau_\beta$$

So the the meromorphic sections τ_α are all equal, and define a meromorphic section τ of L. Clearly

$$(\tau) = \sum_k a_k D_k$$

In conclusion: *a Weil divisor is Cartier iff it is locally principal.* In particular, if X is smooth the notions of Cartier and Weil divisors coincide. More generally, this holds when X is *locally factorial*, i.e., when all local rings are UFDs. (Warning: this is an algebraic property, and not verifiable analytically or at the level of completions.) We can summarise the story so far with the diagram of inclusions of sets (in fact injective homomorphisms of abelian groups):

$$\begin{array}{ccccc} \frac{\mathbb{C}(X)^*}{\mathbb{C}[X]^*} & \hookrightarrow & \sqcup_{L\in Pix(X)}\frac{Mer(L)^*}{\mathbb{C}[X]^*} & & \\ \downarrow\sim & & \downarrow\sim & & \\ \textit{principal divisors} & \hookrightarrow & \textit{Cartier divisors} & \hookrightarrow & \textit{Weil divisors} \end{array}$$

where the disjoint union $\sqcup_L$ is over the group of isomorphism classes of line bundles, the *Picard group* of X. (The Picard *variety*, which we shall mostly avoid, is a subtler object.)

The "traditional" definition of Cartier divisor is essentially as isomorphism class of pairs (L,σ) with L a line bundle and σ a meromorphic section, but reformulated to avoid talking of line bundles, as follows.

Consider the constant sheaf $\mathcal{M}^*$ of nonzero meromorphic functions. This is the sheaf that associates to any (nonempty) open U the abelian group of nonzero meromorphic functions on X (and associates to the empty set the final object in the category of abelian groups, namely the singleton group). This contains as subsheaf the sheaf $\mathcal{O}_X^*$ of units in the sheaf of algebras $\mathcal{O}_X$. I claim that there is a bijective homomorphism between global sections of the quotient sheaf $\mathcal{M}^*/\mathcal{O}_X^*$ and isomorphism class of pairs (L,σ) as above.

Suppose first that a pair (L,σ) is given. Cover X by open sets U_α over which L is trivialised by meromorphic sections σ_α. (That is, σ_α is regular and non-vanishing on U_α.) Then, for each α, there exists a meromorphic function ϕ_α such that

$$\sigma = \phi_\alpha \sigma_\alpha$$

On overlaps $U_\alpha \cap U_\beta$, the meromorphic functions ϕ_α and ϕ_β differ by a regular invertible function. In other words, the ϕ_α define a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}_X^*$. Changing the local trivialisations σ_α does not change this global section, which therefore depends only on (the isomorphism class of) the pair (L,σ).

Conversely, a global section of $\mathcal{M}^*/\mathcal{O}_X^*$ is, by definition, given by an open cover U_α and meromorphic section functions ϕ_α which on overlaps differ by a regular invertible function $\chi_{\alpha,\beta}$. These functions define a 1-cocycle which can be used to construct a line bundle L and a meromorphic section σ.

We have described the bijective maps

(1) from isomorphism class of pairs (L,σ) to Cartier (=locally principal Weil) divisors
(2) between isomorphism class of pairs (L,σ) and global sections of $\mathcal{M}^*/\mathcal{O}_X^*$.

We can in fact give a bijection between isomorphism classes of pairs $(\underline{L},\sigma)$ and Cartier divisors in the language of locally free rank one sheaves (and bypassing line bundles) as follows. The map to Cartier divisors is given as before, and the converse map as follows: given a Weil divisor $D=\sum_i a_i D_i - \sum_j a'_j D'_j$ (with a_i, a'_j positive), define the subsheaf $\mathcal{O}_X(D)$ of the constant sheaf $\mathcal{M}_X$ of meromorphic functions (this latter is a quasi-coherent sheaf) by:

$$\textit{open } U \subset X \mapsto \mathcal{O}_X(\sum_i a_i D_i - \sum_j a'_j D'_j)(U)$$

where $\mathcal{O}_X(\sum_i a_i D_i - \sum_j a'_j D'_j)(U)$ is the space of regular ϕ functions on $U \setminus (\cup_i D_i)$ and such that

(1) ϕ has poles of order at most a_i along D_i, and

(2) ϕ vanishes to order at least a'_j along D'_j.

One checks that the sheaf so defined is locally free of rank one if (iff?) the divisor $\sum_i a_i D_i - \sum_j a'_j D'_j$ is Cartier. The constant function with value 1 is a meromorphic section of the corresponding line bundle, and the corresponding divisor the one we started with. If you check this claim you will see the reason for the apparent reversal of roles between the a_i and a'_j.

Note that if D, D' are Cartier,

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') = \mathcal{O}_X(D + D')$$

When do two Cartier divisors D and D' define the "same" (i.e., isomorphic) line bundles? If this is the case, there is a line bundle L and two meromorphic sections σ, σ' such that $(\sigma) = D$ and $(\sigma') = D'$. Then $\sigma = \phi\sigma'$ for some meromorphic function ϕ, and $D = (\phi) + D'$, or to put it another way:

$$D - D' = (\phi) \text{ is principal}$$

Terminology: We say that two Cartier divisors are *linearly equivalent* if their difference is a principal divisor. This is clearly an equivalence relation.

Summarising the above discussion:

(1) *Picard group* $\sim$ {*Cartier divisors modulo linear equivalence*}

Remark: Consider the exact sequence of sheaves

$$1 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{M}_X^*/\mathcal{O}_X^* \to 1$$

Noting that $H^1(\mathcal{M}_X^*)$ vanishes because $\mathcal{M}_X^*$ is flasque[1], the long exact sequence in cohomology yields the exact sequence of abelian groups:

$$1 \to \{\mathbb{C}(X)^*/\mathbb{C}[X]^*\} \to H^0(\mathcal{M}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to 1$$

Now, $H^0(\mathcal{M}_X^*/\mathcal{O}_X^*)$ is the group of Cartier divisors, going modulo $\{\mathbb{C}(X)^*/\mathbb{C}[X]^*\}$ is rational equivalence, and $H^1(X, \mathcal{O}_X^*)$ is the Picard group, so we have recovered (1).

Notation/Definition: If D is a Cartier divisor, we mean by $\mathcal{O}_X(D)$ the invertible sheaf defined above. Less precisely, we will mean a line bundle L, together with a meromorphic section σ such that $(\sigma) = D$. As noted earlier, the pair (L, σ) is unique up to isomorphism. Even less precisely, we will mean just the line bundle L. We will often confound L and $\underline{L}$ and $\mathcal{O}_X(D)$.

We end by giving the canonical example of a Weil divisor that is not Cartier. Consider the cone in $\mathbb{C}^3$:

$$\{(x, y, z)|x^2 + y^2 = z^2)\}$$

This is an affine variety of dimension 2. Let D be the dimension 1 subvariety defined by the ideal generated by x and $z - y$. One checks that D is reduced and irreducible, but D is not defined by the vanishing of a single regular function.

[1] http://www.math.umn.edu/ garrett/m/algebra/cech.pdf

1.3. **Line bundles and divisors: some easy remarks.** Let $D = \sum_i a_i D_i - \sum_j a'_j D'_j$ be a Cartier divisor, with $a_i, a'_j > 0$.

(1) The line bundle $\mathcal{O}(D)$ is trivial iff D is a principal divisor.
(2) What are the regular sections of $\mathcal{O}_X(D)$ on (all of) X? By definition, these are the meromorphic functions, regular outside $\cup_i D_i$ and with poles of order at most a_i along D_i and vanishing to order at least order a'_j along D'_j.
(3) If D is an *effective divisor*, that is, a positive linear combination of divisors ($D = \sum_i a_i D_i$, with $a_i > 0$), then $H^0(X, \mathcal{O}(D)) \neq 0$, for then any constant function is a global section.
(4) On a complete (which in our context means projective) variety, if D is effective and nonzero, then $\mathcal{O}(D)$ is a nontrivial line bundle.
(5) Suppose D is effective as above, and $\sigma \in H^0(X, \mathcal{O}(D))$ is *any* nonzero (regular) section, then $(\sigma) = \tilde{D} = \sum_{\tilde{i}} a_{\tilde{i}} D_{\tilde{i}}$ with $a_{\tilde{i}} > 0$ and
$$\sum_{\tilde{i}} a_{\tilde{i}} D_{\tilde{i}} - \sum_i a_i D_i$$
is principal. That is, there exists a meromorphic function ϕ with polar divisor $\tilde{D}$ and zero divisor D.

1.4. **Line bundles on smooth projective curves.** Let us see how the theory plays out on a smooth projective curve X of genus g. Since X is smooth, Weil divisors are locally principal, and we will drop the prefix "Cartier". A divisor is just a formal sum with integer coefficients:
$$D = \sum_{x \in X} a_x x$$
where the a_x are integers and nonzero only for finitely many x. Given a divisor D, we define its degree by
$$\textit{degree } D = \sum_{x \in X} a_x$$
Given a meromorphic function ϕ, the meromorphic form $\phi^{-1} d\phi$ has simple poles at the points x such that $a_x \neq 0$, and the residue at such an x is a_x. OTOH, the sum of residues of a meromorphic form is zero, which proves that
$$\textit{degree}(\phi) = 0$$
Thus the degree is an integer-valued function on the Picard group (in fact, a homomorphism *onto* (why?) $\mathbb{Z}$.)

Clearly, if a line bundle has a nonzero regular section, its degree is nonnegative. If such a line bundle has degree zero, it has a nowhere vanishing regular section and is therefore trivial.

For curves of arbitrary genus it is difficult to proceed without more machinery. Let us consider line bundles on the projective line $\mathbb{P}^1$. Fix a point

x_∞; its complement is isomorphic to the affine line $\mathbb{C}$. Fixing this isomorphism amounts to giving a meromorphic function z with one simple pole at x_∞. Suppose given a divisor $D = \sum_i a_i x_i - \sum_j a_j x_j + a_\infty x_\infty$, where the x_i, x_j are distinct from each other and from x_∞, and $a_i > 0, a_j > 0$. Set

$$\phi = \prod_i (z - z(x_i))^{a_i} \prod_j (z - z(x_j))^{-a_j}$$

Clearly $(\phi) = \sum_i a_i x_i - \sum_j a_j x_j + (\sum_j a_j - \sum_i a_i) x_\infty$, which shows that D is principal iff its degree is zero. We can now conclude that

$$\text{Picard group of } \mathbb{P}^1 \underset{degree}{\rightarrow} \mathbb{Z}$$

is an isomorphism. Given $l \in \mathbb{Z}$, there is up to isomorphism one line bundle of degree l, which we can take to be $\mathcal{O}(l x_\infty)$. Regular sections of $\mathcal{O}(l x_\infty)$ are polynomial functions on $\mathbb{C}$ with

(1) a zero of order at least $-l$ at ∞ if $l < 0$,
(2) regular at ∞ if $l = 0$, and
(3) a pole of order at most l at ∞ if $l > 0$,

It follows that $H^0(\mathcal{O}(l x_\infty)) = 0 \; if \; l < 0$, $H^0(\mathcal{O}(l x_\infty)) = \mathbb{C} \; if \; l = 0$, and if $l > 0$

$$H^0(\mathcal{O}(l x_\infty)) = \text{polynomials of degree} \leq l \text{ if } l > 0. \tag{2}$$

In particular if L is a ('the') line bundle of degree l,

$$dim\ H^0(\mathbb{P}^1, L) = l + 1$$

2. Linear series, Maps to projective space

Abuse of notation ahead: From now on we shall cease differentiating between a vector bundle E and the corresponding sheaf of sections $\underline{E}$, except when the occasion seems to call for it.

2.1. **Preliminaries on Grassmannians.** Given a finite-dimensional vector space V, we will mean by the *projective space* $\mathbb{P}(V)$ the variety which parametrises one-dimensional quotients of V. More generally, consider the Grassmannian $Gr_k(V)$ of k dimensional quotients of V, where k is an integer, $0 < k < dim\ V$. Any point $V \to Q \to 0$ with $dim\ Q = k$ of $Gr_k(V)$ determines an exact sequence of vector spaces $0 \to S \to V \to Q \to 0$, with $dim\ Q = k$. Two such quotients are identified if there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle \sim}\downarrow & & \\ 0 & \longrightarrow & S' & \longrightarrow & V & \longrightarrow & Q' & \longrightarrow & 0, \end{array}$$

in other words, if the kernels S, S' are the same. Thus $Gr_k(V)$ can be identified with the set of $(dim\ V - k)$-dimensional subspaces of V. Grothendieck adopted the quotient definition (which requires the above fuss with identifying quotents) for good reason, and (following Lazarsfeld) we will stick to it. On $Gr_k(V)$ the vector spaces Q and S together build up tautological vector bundles $\mathcal{Q}$ of rank k and $\mathcal{S}$ of rank $(dim\ V - k)$ as quotient and (respectively) sub of the trivial bundle with fibre V. In terms of the corresponding locally free sheaves, we write:

$$0 \to \underline{\mathcal{S}} \to V \otimes_{\mathbb{C}} \mathcal{O}_{Gr_k(V)} \to \underline{\mathcal{Q}} \to 0$$

Allowing for abuse of notation, the above sequence could be written:

$$0 \to \mathcal{S} \to V \otimes_{\mathbb{C}} \mathcal{O}_{Gr_k(V)} \to \mathcal{Q} \to 0$$

<u>Remark:</u> If V is a finite-dimensional vector space, the trivial vector bundle $X \times V \to X$, which we could denote by (say) V_X, has the associated locally free sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_X$. The latter notation is standard, but there is no standard notation for the geometric vector bundle. So you can think of the above sequence as standing in for the exact sequence of vector bundles $0 \to \mathcal{S} \to V_X \to \mathcal{Q} \to 0$.

It is a **fact** that the tangent bundle of $Gr_k(V)$ is $Hom(\mathcal{S}, \mathcal{Q}) = \check{\mathcal{S}} \otimes_{\mathbb{C}} \mathcal{Q}$.

The variety $Gr_k(V)$ has the following important property. Given a rank r vector bundle $\mathcal{Q}_X$ on X, expressed as a quotient

$$V \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{Q}_X \to 0$$

there is a unique morphism $\psi : X \to Gr_k(V)$ and an unique isomorphism $\psi^* \mathcal{Q} \to \mathcal{Q}_X$ which makes the following commutative:

$$\begin{array}{ccccc} V \otimes_{\mathbb{C}} \mathcal{O}_X & \longrightarrow & \psi^* \mathcal{Q} & \longrightarrow & 0 \\ {\scriptstyle =}\big\downarrow & & {\scriptstyle \sim}\big\downarrow & & \\ V \otimes_{\mathbb{C}} \mathcal{O}_X & \longrightarrow & \mathcal{Q}_X & \longrightarrow & 0 \end{array}$$

Set-theoretically the map ψ is described as follows. Let $x \in X$; the surjective map of bundles $V \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{Q}_X \to 0$ yields the surjective map of vector spaces:

$$V \to (\mathcal{Q}_X)_x \to 0$$

which represents the point $\psi(x) \in Gr_k(V)$. We will prove below that ψ is a morphism in the case that interests us, that of projective spaces.

2.2. **Projective spaces.** Specialise now to $k = 1$. On $\mathbb{P}(V)$ the tautological quotient line bundle is conventionally denoted by $\mathcal{O}(1)$ (and its dual by $\mathcal{O}(-1)$). From the above remarks, we see that it sits in an exact sequence

$$0 \to \Omega_{\mathbb{P}(V)} \otimes_{\mathbb{C}} \mathcal{O}(1) \to V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}(1) \to 0$$

where, for a smooth variety X, Ω_X denotes cotangent bundle. (We have used **fact** from above.) Note that this gives a formula for the canonical bundle: $\kappa = (det\ V) \otimes_{\mathbb{C}} \mathcal{O}(-dim\ V)$. It also follows (using the vanishing of $H^0(\Omega_{\mathbb{P}(V)} \otimes_{\mathbb{C}} \mathcal{O}(1))$ and $H^0(\Omega_{\mathbb{P}(V)} \otimes_{\mathbb{C}} \mathcal{O}(1))$, special cases of a vanishing theorem due to Bott) that the map

$$V = H^0(\mathbb{P}(V), V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)}) \to H^0(\mathbb{P}(V), \mathcal{O}(1))$$

is an isomorphism. Of course, the injectivity of the above map is clear (given a nonzero vector $v \in V$, its image under a surjection $V \to Q \to 0$ is nonzero outside a linear hyperplane), and we will outline below an argument for the surjectivity.

On occasion, we will need to deal with one-dimensional *sub*spaces of a vector space. For the moment consider $\check{V}$, the dual of V. The corresponding variety will be denoted $\mathbb{P}_{sub}(\check{V})$; clearly $\mathbb{P}_{sub}(\check{V}) = \mathbb{P}(V)$. This *avatar* of projective space is easier to visualise. The one-dimensional subspaces fit together into a "tautological" line sub-bundle of the trivial vector bundle with fibre $\check{V}$; let us denote this line bundle by $\tilde{\mathcal{O}}(-1)$ and its dual by $\tilde{\mathcal{O}}(1)$. The total space of $\tilde{\mathcal{O}}(-1)$ is the canonical example of a blow-up (of $\check{V}$ at the origin). We have

$$0 \to \tilde{\mathcal{O}}(-1) \to \check{V} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}_{sub}(\check{V})} \to \Theta_{\mathbb{P}_{sub}(\check{V})} \otimes \tilde{\mathcal{O}}(-1) \to 0$$

where Θ defines the tangent bundle. Comparing exact sequences we see that under the identification $\mathbb{P}_{sub}(\check{V}) = \mathbb{P}(V)$ the line bundles $\tilde{\mathcal{O}}(\pm 1)$ coincide with $\mathcal{O}(\pm 1)$. (This gives a direct proof of the surjectivity of $V \to H^0(\mathbb{P}(V), \mathcal{O}(1))$. For, a section of $\mathcal{O}(1)$ gives a regular function on the total space of $\mathcal{O}(-1) = \tilde{\mathcal{O}}(-1)$, homogeneous of degree 1, which in turn gives a regular function on $\check{V} \setminus \{0\}$ homogeneous of degree 1. This has to extend across the origin, and determines an element of V.)

Homogeneous coordinates: Given a point $V \to Q \to 0$ of $\mathbb{P}(V)$, choosing an ismorphism $Q \sim \mathbb{C}$, we get a nonzero element $\check{v} \in \check{V}$. Clearly $\check{v}$ spans the corresponding one-dimensional subspace of $\mathbb{P}_{sub}(\check{V})$. Given any such nonzero $\check{v}$ we say that the homogeneous coordinate of the point is $[\check{v}]$.

Suppose given a line bundle bundle $\mathcal{Q}_X$ on X, expressed as a quotient

$$V \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{Q}_X \to 0$$

let us prove that the corresponding map $\psi : X \to \mathbb{P}(V)$ is a morphism. Let $V = \ell \oplus V'$ where ℓ is a one-dimensional subspace. This defines an open set U_ℓ in $\mathbb{P}(V)$ such that the composite map

$$\ell \otimes_{\mathbb{C}} \mathcal{O}_{U_\ell} (\hookrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{U_\ell}) \to \mathcal{O}(1)$$

is an isomorphism. (Note that the open set U_ℓ depends only on ℓ and not on the choice of supplement V'.) This gives a map of bundles on U_ℓ:

$$V' \otimes_{\mathbb{C}} \mathcal{O}_{U_\ell} \to \mathcal{O}(1) \sim \ell \otimes_{\mathbb{C}} \mathcal{O}_{U\ell} \ .$$

The structure of variety on $\mathbb{P}(V)$ is *defined* so that the induced (bijective) map $\phi_\ell : U_\ell \to Hom(V', \ell)$ is an isomorphism. The inverse image of U_ℓ by

ψ is clearly (the open set) $\tilde{U}_\ell \subset X$ such that the induced map

$$\ell \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{U}_\ell}(\hookrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{U}_\ell}) \to \mathcal{Q}_X \to 0$$

is an isomorphism. This gives a map of bundles on $\tilde{U}_\ell$:

$$V' \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{U}_\ell} \to \mathcal{Q}_X \sim \ell \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{U}\ell} \, .$$

which in turn yields a morphism

$$\tilde{U}_\ell \underset{\tilde{\psi}_\ell}{\to} Hom(V', \ell)$$

Clearly $\tilde{\psi}_\ell = \phi_\ell \circ \psi|_{\tilde{U}_\ell}$, or equivalently

$$\psi|_{\tilde{U}_\ell} = \phi_\ell^{-1} \circ \tilde{\psi}_\ell$$

which shows that $\psi|_{\tilde{U}_\ell}$ is a morphism. Since the sets U_ℓ cover $\mathbb{P}(V)$ we are done.

From now on, we will (unless otherwise flagged) consider projective varieties.

If X is projective and L is a line bundle then $H^0(X, L)$ is a finite-dimensional vector space. Any nonzero section defines an effective divisor, and two sections define the same divisor iff one is a nonzero scalar multiple of the other. This has the following consequence: given a Cartier divisor D on a projective variety, the set

$$\{\tilde{D} | \tilde{D} \text{ is effective and linearly equivalent to } D\}$$

can naturally be identified with the projective space $\mathbb{P}_{sub}(H^0(X, \mathcal{O}_X(D)))$ of one-dimensional subspaces of $H^0(X, \mathcal{O}_X(D))$. (Of course, it could happen that $\mathcal{O}_X(D)$ has no nonzero regular sections, in which case $|D|$ is empty. If D itself is effective, it will represent a point of the projective space.)

By a *linear series*, we will mean (interchangeably) a nonzero subspace $V \subset H^0(X, L)$ or the set $|V|$ of effective divisors corresponding to such a subspace V. Note that the set $|V|$ can be identified with $\mathbb{P}_{sub}(V) \subset \mathbb{P}_{sub}(H^0(X, L))$. If $V = H^0(X, L)$ we will say that the linear series is *complete*. If D is a Cartier divisor, we denote by $|D|$ the complete linear series associated to $\mathcal{O}_X(D)$.

Given a linear series V, its *base locus* $Bs(|V|)$ is the (Zariski)-closed set of points where all the nonzero sections in V vanish. If $L = \mathcal{O}(D)$ and V is the corresponding complete linear series, we will also denote this by $Bs(|L|) = Bs(|D|)$. The notation signifies that the base locus comes with a natural scheme structure, and one can in fact talk of the *base scheme* defined by the sheaf of ideals which is the image of the sheaf map ("evaluate $\sigma \in V$ at $x \in X$ to get $\sigma(x) \in L_x$, compose with the map $L_x \otimes L_x^{-1} \to \mathbb{C}$"):

$$V \otimes_{\mathbb{C}} L^{-1} \to \mathcal{O}_X$$

We will not deal with the base scheme.

We say that the linear series $|V|$ is *(base-point) free* if its base locus is empty. In other words, $|V|$ is free if for every $x \in X$, there is a section belonging to V which is nonvanishing at x. Given a line bundle L (or a divisor D), we say that it is free ("generated by global sections") if the corresponding complete series is free.

Outside the base locus B of V we have the evaluation morphism e("evaluate $\sigma \in V$ at $x \in X$ to get $\sigma(x) \in L_x$"):

$$V \otimes_{\mathbb{C}} \mathcal{O}_X \underset{e}{\rightarrow} L \rightarrow 0$$

This yields a morphism $\phi_{|V|} : X \setminus B \rightarrow \mathbb{P}(V)$ and a commutative diagram:

$$\begin{array}{ccccc} V \otimes_{\mathbb{C}} \mathcal{O}_X & \longrightarrow & \psi^*\mathcal{O}(1) & \longrightarrow & 0 \\ {\scriptstyle =}\big\downarrow & & {\scriptstyle \sim}\big\downarrow & & \\ V \otimes_{\mathbb{C}} \mathcal{O}_X & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Often we think of $\phi_{|V|}$ as a rational map $\phi_{|V|} : X \dashrightarrow \mathbb{P}(V)$.

Let us make this explicit. Let $dim\ V = n$. Choose a basis of sections $\sigma_1, \ldots, \sigma_n \in V$, and let $\check{\sigma}_i \in \check{V}$ be the dual basis. Let $U_i \subset X$ be the open set

$$U_i = \{x \in X | \sigma_i(x) \neq 0\}$$

On U_1 we have regular functions $x_1 \equiv 1, x_2, \ldots, x_n$ such that

$$e(\sum_i a_i \sigma_i) = (\sum_i a_i x_i)\sigma_1(x)$$

which shows that the corresponding map $U_1 \rightarrow \mathbb{P}_{sub}(\check{V})$ is given by

$$x \mapsto [\sum_i x_i \check{\sigma}_i]$$

Suppose that X is a normal projective variety, L a line bundle, and $V \subset H^0(X, L)$ a linear series, For simplicity we take $|L|$ to be base-point free. How are the morphisms induced by V and the complete linear series $H^0(X, L)$ related? Given a one-dimensional quotient $H^0(X, L) \rightarrow Q \rightarrow 0$, the composite map $V \subset H^0(X, L) \rightarrow Q$ will give a point of $\mathbb{P}(V)$ iff this map is nonzero. So let us identify the set of quotients $H^0(X, L) \rightarrow Q \rightarrow 0$ such that the composite map $V \subset H^0(X, L) \rightarrow Q$ *is* zero. This happens iff the dual map $\check{Q} \subset H^0(X, L)^\vee \rightarrow \check{V}$ is zero. That is iff, under the identification

$$\mathbb{P}(H^0(X, L)) = \mathbb{P}_{sub}(H^0(X, L)^\vee)$$

the point $H^0(X, L) \rightarrow Q \rightarrow 0$ belongs to $\mathbb{P}_{sub}(V^\perp)$, where $V^\perp \subset H^0(X, L)^\vee$ is the annihilator of V. Thus we have a morphism ("projection")

$$\pi : \mathbb{P}(H^0(X, L)) \setminus \mathbb{P}_{sub}(V^\perp) \rightarrow \mathbb{P}(V)$$

and

1. $|V|$ is free iff $\mathbb{P}_{sub}(V^\perp) \cap\ image(\phi_L) = \emptyset$,
2. in general $Bs(|V|) = \phi_L^{-1}(\mathbb{P}_{sub}(V) \cap\ image(\phi_L))$, and
3. on $X \setminus Bs(|V|)$, we have $\phi_{|V|} = \pi \circ \phi_{|L|}$.

Let X be a projective variety, and suppose given a map $\phi : X \to \mathbb{P}(V)$ for some vector space V; set $L \equiv \phi^*\mathcal{O}(1)$. This yields a map

$$V = H^0(\mathbb{P}(V), \mathcal{O}(1)) \to H^0(X, \phi^*\mathcal{O}(1)) = H^0(X, L)$$

Since X is projective (and irreducible), its image in $\mathbb{P}(V)$ is a closed (and irreducible) set and hence a (reduced) subvariety $X' \subset \mathbb{P}(V)$. In fact, since X is reduced, the morphism ϕ factors through X' (exercise). We write: $\pi = \iota \circ \pi'$ where

$$\phi' : X \to X' \ \ \textit{and}\ \iota : X' \to \mathbb{P}(V)$$

The pull-back map on sections also factors:

$$V = H^0(\mathbb{P}(V), \mathcal{O}(1)) \underset{\iota^*}{\to} H^0(X', \mathcal{O}(1)|_{X'}) \underset{\phi'^*}{\to} H^0(X, \phi'^*\mathcal{O}(1)|_{X'}) = H^0(X, L)$$

Since the map ϕ' is surjective, the map ϕ'^* is injective (locally on X') on functions, and therefore on sections of $\mathcal{O}(1)|_{X'}$ (by local triviality). The map ι^* is injective iff X' is not contained in any hyperplane in $\mathbb{P}(V)$. (In this case one sometimes says that X' s a *nondegenerate subvariety* of projective space.) To summarise the discussion so far:

Let X be a projective variety, and suppose given a map $\phi : X \to \mathbb{P}(V)$ for some vector space V, with nondegenerate image. Then $V \subset H^0(X, L)$ is a base-point free linear series and $\phi = \phi_{|V|}$.

For V to be the *complete linear series* $H^0(X, L)$, we need both ι^* and ϕ^* to be surjective. More on this later.

Finally, we note the following. If X is nonsingular, $\tilde{B}$ a closed subset of codimension ≥ 2 and $\phi : X \setminus \tilde{B} \to \mathbb{P}(V)$ a morphism whose image is nondegenerate, then

(1) $\phi^*\mathcal{O}(1)$ extends uniquely as a line bundle L on X, and

$$V = H^0(\mathbb{P}, \mathcal{O}(1)) \subset H^0(X \setminus \tilde{B}, \mathcal{O}(1)) = H^0(X, L),$$

(2) $B \equiv Bs(|V|) \subset \tilde{B}$, and $\phi_{|V|} : X \setminus B$ extends ϕ.

2.3. **Example: The projective line as conic.** First some preliminary linear algebra. Given an exact sequence of vector spaces:

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

then, for any $k \geq 2$, the induced map $S^kV_2 \to S^kV_3$ is onto, and has kernel (isomorphic to) $V_1 \otimes_{\mathbb{C}} S^{k-1}V_2$.

Let now W be a two dimensional vector space, and set $X = \mathbb{P}(W)$, the corresponding projective space. We have the tautological sequence

$$0 \to \Omega \otimes_{\mathcal{O}_X} \mathcal{O}(1) \to W \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}(1) \to 0$$

This induces:

$$0 \to S^{k-1}W \otimes_{\mathbb{C}} \Omega \otimes_{\mathcal{O}_X} \mathcal{O}(1) \to S^kW \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}(k) \to 0$$

which yields (using vanishing theorems as we did earlier, or using (2)):

$$S^kW = H^0(X, \mathcal{O}(k)),\ k \geq 1$$

Consider the complete free linear series $H^0(X, \mathcal{O}(k))$. This yields maps:

$$\phi_k \equiv \phi_{|\mathcal{O}(k)|} : X \to \mathbb{P}(H^0(\mathbb{P}(W), \mathcal{O}(k))) = \mathbb{P}(S^kW)$$

In terms of homogeneous coordinates

$$\phi_k([\tilde{w}]) = [\tilde{w}^k]$$

In the case $k = 2$, the image is a curve in the projective plane $\mathbb{P}(S^2W)$. By definition $\mathcal{O}_{\mathbb{P}(S^2W)}(1)$ pulls back to $\mathcal{O}_X(2)$, so the image is a curve of degree 2. (We are anticipating some interesection theory, which is yet to come.)

Consider the map $B : S^2W \times S^2W \to (det\ W)^2$ given by

$$B(\sum_i u_i v_i, \sum_j u'_j v'_j) = \sum_{i,j} (u_i \wedge u'_j)(v_i \wedge v'_j) + (u_i \wedge v'_j)(v_i \wedge u'_j)$$

This is clearly bilinear and symmetric, an so defines a homogeneous polynomial of degree 2 on S^2W with values in the line (i.e., one-dimensional complex vector space) $(det\ W)^2$. This can be thought of as a section of

$$H^0(\mathbb{P}(S^2W), \mathcal{O}_{\mathbb{P}(S^2W)}(2) \otimes_{\mathbb{C}} (det\ W)^2) = S^2(S^2W) \otimes_{\mathbb{C}} (det\ W)^2$$

whose zero set is precisely the image of X under ϕ_2.

3. Intersection classes; intersection numbers

A comprehensive development of the theory is in Fulton's book. The beginner should read the treatment of intersection theory on surfaces in Hartshorne's *Algebraic Geometry*. Both books treat intersection theory as defining a product on the Chow ring of algebraic cycles (formal linear combination of irreducible subvarieties, modulo rational equivalence). In codimension one, this is the group of Weil divisors modulo principal divisors. Chern classes (see below) are defined to take values in the Chow ring. For a nice summary (of the theory on smooth varieties), see Hartshorne's Appendix. See also *3264 & All That: Intersection Theory in Algebraic Geometry*, Eisenbud and Harris.[2])

We follow Lazarsfeld, who has a more topological approach that suffices for complex algebraic varieties.

Topologically, a line bundle L_{top} is classified up to isomorphism by its Chern class, an element $c_1(L_{top}) \in H^2(X, \mathbb{Z})$. This is best defined in terms of Čech cohomology. Consider the exact sequence of sheaves in the analytic (not Zariski) topology:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{cont} \underset{f \mapsto e^{2\pi i f}}{\to} \mathcal{O}^*_{cont} \to 0$$

where $\mathcal{O}_{cont}$ is the (additive) sheaf of continuous complex-valued functions and $\mathcal{O}^*_{cont}$ the (mutliplicative) sheaf of invertible continuous complex-valued

[2] http://scholar.harvard.edu/files/joeharris/files/000-final-3264.pdf

functions. If X is connected and paracompact (as irreducible varieties certainly are in the analytic topology), the higher cohomologies of $\mathcal{O}_{cont}$ vanish, and we get

$$H^1(X, \mathcal{O}^*_{cont}) \sim H^2(X, \underline{\mathbb{Z}})$$

The cohomology group on the left classifies isomorphism classes of continuous line bundles L_{top}, and $c_1(L_{top})$ is (*up to a sign change*) the corresponding element of $H^2(X, \underline{\mathbb{Z}})$.

Instead of working with continuous functions, let us work with the sheaf of analytic functions, again in the analytic topology:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{an} \underset{f \mapsto e^{2\pi i f}}{\to} \mathcal{O}^*_{an} \to 0$$

This yields:

$$\{1\} \to \frac{H^1(X, \mathcal{O}_{an})}{H^1(X, \underline{\mathbb{Z}})} \to H^1(X, \mathcal{O}^*_{an}) \to H^2(X, \underline{\mathbb{Z}}) \to \{1\}$$

The cohomology group $H^1(X, \mathcal{O}^*_{an})$ classifies isomorphism classes of holomorphic line bundles. The map $H^1(X, \mathcal{O}^*_{an}) \to H^2(X, \underline{\mathbb{Z}})$ clearly factors as:

$$H^1(X, \mathcal{O}^*_{an}) \to H^1(X, \mathcal{O}^*_{cont}) \to H^2(X, \underline{\mathbb{Z}})$$

which shows that topologically trivial line bundles are parametrised by (the Jacobian)

$$\frac{H^1(X, \mathcal{O}_{an})}{H^1(X, \underline{\mathbb{Z}})}$$

Finally, on an algebraic variety, *algebraic* line bundles are classified by $H^1(X, \mathcal{O}^*)$ where now $\mathcal{O}^*$ is the Zariski sheaf of invertible regular functions. It is not entirely obvious how to compare the cohomology groups $H^1(X, \mathcal{O}^*)$ and $H^1(X, \mathcal{O}^*_{an})$, but GAGA assures that on a projective variety analytic and algebraic vector bundles "are the same". To summarise, to an (algebraic) line bundle on a projective variety, one can associate the Chern class $c_1(L) \in H^2(X, \mathbb{Z})$.

We have, given line bundles $L,\ L'$,

$$c_1(L \otimes_{\mathbb{C}} L') = c_1(L) \wedge c_1(L')$$

where the product is taken in the cohomology ring. In fact, these (and higher Chern classes which are relevant for vector bundles of arbitrary rank) are *even* degree classes, and products are taken in the commutative ring $H^*_{even}(X, \mathbb{Z}) \equiv \sum_i H^{2i}(X, \mathbb{Z})$.

Not every topological line bundle admits an algebraic structure. If X is smooth and projective, the Lefshetz Theorem on $(1,1)$-classes gives a necessary and sufficient condition: given an algebraic L the image of $c_1(L)$ in $H^2(X, \mathbb{R}) \subset H^2(X, \mathbb{C}) = H^{(2,0)}(X, \mathbb{C}) \oplus H^{(1,1)}(X, \mathbb{C}) \oplus H^{(0,2)}(X, \mathbb{C})$ lies in the factor $H^{(1,1)}(X, \mathbb{C})$, and every such (integral) class arises this way.

From now on, varieties are projective unless otherwise flagged.

Let Y be a complex projective variety of dimension k. "By a theorem of Lojasiewicz [1964] (see Hironaka [1975]), Y admits a finite triangulation in which the singular locus is a subcomplex. Since the singularities of Y occur in real codimension 2, the sum of the simplices of dimension $2k$ in the triangulation is a cycle, called the fundamental cycle of Y ; the class of this cycle is called the fundamental class of Y and is again denoted by $[Y] \in H_{2k}(Y;\mathbb{Z})$" (from *3264 & All That: Intersection Theory in Algebraic Geometry.*)

Having finished with our quote, let us revert to considering a projective variety X. Given a subvariety V, consider the image of the fundamental class $[V]$ under the map $H_{2k}(V;\mathbb{Z}) \to H_{2k}(X;\mathbb{Z})$ induced by the inclusion of V in X; by abuse of notation, we will continue to denote this by $[V] \in H_{2k}(X;\mathbb{Z})$.

Definition: More generally, given a subscheme V of pure dimension k (a term we will explain in a minute), we will set

$$[V] = \sum_i l_i[V_i]$$

Here the sum is taken over the components V_i of the reduced subscheme V_{red}. To say that V has pure dimension k means that each component V_i has dimension k. (Warning: if V has embedded components, the definition is disputed. See https://mathoverflow.net/questions/30495/pure-dimensional-and-embedded-components.) The integers l_i are defined by

$$l_i = length_{\mathcal{O}_{V_i}} \mathcal{O}_V$$

If V is a subvariety of dimension k and $L_i,\ i = 1,\dots,k$ are line bundles, we use the pairing $H^{2k} \times H_{2k} \to \mathbb{Z}$ to define the intersection number

$$< c_1(L_1) \wedge \dots \wedge c_k(L_k), [V] > \in\ \mathbb{Z}$$

We extend this by linearity to the case when V is a subscheme. If $L_i = \mathcal{O}(D_i)$, we also use the notations:

$$\int_V D_1 \cdot \cdots \cdot D_k \tag{3}$$

or

$$(D_1 \cdot \cdots \cdot D_k \cdot V) \tag{4}$$

for the above intersection number. Clearly $(D_1 \cdot \cdots \cdot D_k \cdot V)$ depends only on the rational equivalence classes of the D_i's, and is mutilinear and symmetric in the D_is.

The last two notations are very suggestive and indicate two other ways of viewing these numbers. The expression (3) says that the intersection number can be computed as an integral. Each of the Chern classes is represented by a closed (1,1)-form on X^{reg}, their product is a (k,k)-form, which can be integrated over V^{reg} to yield the intersection number. (Since V^{reg} is noncompact if V is singular, this is not obvious.) The expression (4) signals a way of computing intersection numbers which is very important:

The intersection theory black box.

Let $D_1, \ldots, D_n$ be Cartier divisors:

$$D_i = \sum_{j_i} a_{i,j_i} D_{i,j_i}$$

with $a_{i,j_i} \in \mathbb{Z}$ and the D_{i,j_i} reduced, irreducible codimension one subvarieties. Then

$$(D_1 \cdot \cdots \cdot D_n \cdot X) = \sum_{j_1,\ldots,j_n} a_{1,j_1} \ldots a_{n,j_n} \ \#\{D_{1,j_1} \cap \cdots \cap D_{n,j_n}\}$$

provided that at each of the intersection points p in the sum,

- each of the the relevant codimension one subvarieties is regular, as is X, and
- one can find analytic local coordinates $z_1, \ldots, z_n$ on a neighbourhood $U \subset X^{reg}$ such that for all i, $z_i(p) = 0$ and $D_{i,j_i} \cap U = \{z_i = 0\}$.

Given a subvariety V of dimension k, then $(D_1 \cdot \cdots \cdot D_k \cdot V)$ can be computed by replacing each D_i by a linearly equivalent divisor D_i' such its support does not contain V, and intersecting the restrictions $D_i'|_V$. If D_n is an irreducible, reduced, codimension one subvariety that is locally principal,

$$(D_1 \cdot \cdots \cdot D_n \cdot X) = (D_1 \cdot \cdots \cdot D_{n-1} \cdot D_n)$$

where on the right-hand side we are intersecting $n-1$ divisors on the subvariety D_n.

Note:

(1) In particular, if L is a line bundle on X and $C \subset X$ is a curve, $deg\ L|_C \equiv c_1(L) . C \equiv < c_1(L), [C] >$ is the pairing of the cohomology class $c_1(L)$ and the homology class $[C]$. If C is smooth, this coincides with the degree of L restricted to C as defined earlier.

(2) If X is a projective surface, and C, C' irreducible Cartier divisors intersecting transversally (only) at regular points of X,

$$\#\{C \cap C'\} = deg\ \mathcal{O}(C)|_{C'} = deg\ \mathcal{O}(C')|_C$$

(If C and C' are not assumed locally principal, is this still true? I am not sure.)

3.1. **Numerical equivalence.** Let X be a projective variety. We have seen that

$$\frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})} < Pic(X)$$

with the quotient mapping (via the Chern class map) injectively into $H^2(X, \mathbb{Z})$; let us denote by $H^2(X, \mathbb{Z})_{alg}$ the image. The image $H^2(X, \mathbb{Z})_{alg}$ contains the torsion classes in $H^2(X, \mathbb{Z})$. If X is smooth, the subgroup $\frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$ is naturally a complex torus and a projective variety (in other words, an *abelian variety*).

On a general variety X, we say that two line bundles L_1 and L_2 are

(1) *algebraically equivalent* if there is a connected variety T, a line bundle $\mathcal{L}$ on $T \times X$, and points $t_1, t_2 \in T$ such that $\mathcal{L}|_{\{t_j\}\times X} = L_j,\ j = 1, 2$.
(2) *homologically equivalent* if $c_1(L_1) = c_1(L_2)$
(3) *numerically equivalent* if $< c_1(L_1), C >=< c_1(L_2), C >$ for every irreducible curve $C \subset X$.

These are all equivalence relations[3]. We have

$$\text{Algebraic equivalence} \underset{A}{\Longrightarrow} \text{homological equivalence} \underset{B}{\Longrightarrow} \text{numerical equivalence.}$$

At least for smooth projective varieties, the implication A can be reversed. Important fact: going modulo numerical equivalence kills *only the torsion in* $H^2(X, \mathbb{Z})_{alg}$.

Summarising:

$$Pic(X)/\{\text{hom. eq. } (= \text{alg. eq. in smooth projective } X)\} \underset{L\mapsto c_1(L)}{\sim} H^2(X,\mathbb{Z})_{alg}$$
$$\underset{\text{num. eq.}}{\rightarrow} H^2(X,\mathbb{Z})_{alg}/H^2(X,\mathbb{Z})^{tor} \equiv N^1(X)$$

where, following Lazarsfeld, we define[4] the *Neron-Severi group* $N^1(X)$ to be $Pic(X)/\{\text{numerical equivalence}\}$.

Note that the Neron-Severi group is a free abelian group of *finite rank less than or equal to the second Betti number of* X. The rank of the Neron-Severi group is called the Picard number; we denote it by $\rho(X)$.

Intersection numbers are unchanged under numerical equivalence. To see this note that by definition these depend only on the Chern classes of the line bundles in question; note next that a torsion class must have zero intersection number with anything else.

4. Asymptotic Riemann-Roch

The (Hirzebruch-)Riemann-Roch Theorem computes the Euler characteristic of a coherent sheaf in terms of its Chern Classes. A consequence of this theorem (or, rather, its analogues for singular varieties) is the following, which we will use:

Theorem 4.1. *Let X be an n-dimensional projective variety, $\mathcal{F}$ a coherent sheaf, D a Cartier divisor, $L = \mathcal{O}(D)$. Then there is a polynomial with rational coefficients which gives at integral values the function $m \mapsto \chi(\mathcal{F} \otimes L^m)$, and*

$$\chi(\mathcal{F} \otimes L^m) = (rank\ \mathcal{F})\frac{D^n}{n!}m^n + \text{ terms of order } m^{n-1} \text{ or lower}$$

---

[3]See https://math.stackexchange.com/questions/1057281/algebraic-equivalence-of-line-bundles for a clever proof in the first case.

[4]This is slightly unconventional; the usual definition identifies $NS(X)$ with $H^2(X, \mathbb{Z})_{alg}$.

A coherent sheaf $\mathcal{F}$ on an irreducible (not necessarily projective) scheme is free on a (Zariski) open, dense set. By *rank* $\mathcal{F}$ we mean the rank r of this free sheaf; the rank is zero iff the support of $\mathcal{F}$ is a proper (closed) subset. The term D^n refers to the self-intersection

$$D^n = (\underbrace{D \cdot \cdots \cdot D}_{n \ times})$$

We will give a proof of the Theorem, stopping short of pinning down the dependence on the self-intersection of D. We follow the treatment of O. Debarre's "Higher-Dimensional Algebraic Geometry" where a more general result is proved.

Proof. Recall that associated to any coherent sheaf $\mathcal{F}$ on an (irreducible, reduced) variety is its torsion subsheaf $\mathcal{F}_{tor}$, whose support Y is a proper closed subscheme (which need not be reduced or irreducible.) We have

$$0 \to \mathcal{F}_{tor} \to \mathcal{F} \to \mathcal{F}_{tf} \to 0$$

with $\mathcal{F}_{tf}$ torsion-free. Clearly, $rank\ \mathcal{F} = rank\ \mathcal{F}_{tf}$. We will now use a very useful result from EGA III.3.1 which implies: *a coherent sheaf $\mathcal{G}$ on a projective scheme Y has a filtration with successive quotients torsion-free on reduced, irreducible sub-varieties of Y.* Applying this to $\mathcal{F}_{tor}$, and by induction on n, we see that $\chi(\mathcal{F}_{tor} \otimes L^m)$ has order at most m^{n-1}, so we can assume that $\mathcal{F}$ itself is torsion-free.

Let $\mathcal{F} \to \mathcal{O}^r$ be an isomorphism on an open $U \subset X$. We claim that the morphism $\mathcal{F} \to \mathcal{O}^r$ extends to an injection $\mathcal{F} \to \mathcal{K}_X^r$ on X where $\mathcal{K}$ is the quasi-coherent (constant) sheaf of meromorphic functions on X. Let $\mathcal{G} = \mathcal{F} \cap \mathcal{O}^r$. The quotients $\mathcal{F}/\mathcal{G}$ and $\mathcal{O}^r/\mathcal{G}$ are both supported on lower-dimensional subschemes of X, so as above we argue that

$$\chi(\mathcal{F} \otimes L^m) \sim \chi(\mathcal{G} \otimes L^m) \sim \chi(\mathcal{O}^r \otimes L^m)$$

where $\sim$ indicates an equality up to terms of order m^{n-1} or lower.

So it suffices to treat the case $\mathcal{F} = \mathcal{O}$.

Let $D = D_1 - D_2$ where D_1 and D_2 are positive integral combinations of irreducible codimension one subvarieties. Then $\mathcal{O}(-D_j)$, $j = 1, 2$, defined earlier in these notes, are sheaves of ideals (not necessarily invertible, unless both D_j are Cartier); to emphasize this, let us write $\mathcal{O}(-D_j) = \mathcal{I}_j$. Let D_j denote the corresponding subschemes. We have exact sequences:

$$0 \to \mathcal{I}_j \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0, \quad i = 1, 2$$

Tensoring by the locally free sheaf $\mathcal{O}(lD)$ these remain exact, so we get:

$$0 \to \mathcal{O}(lD) \otimes \mathcal{I}_j \to \mathcal{O}(lD) \to \mathcal{O}(lD)|_{D_j} \to 0$$

Take $l = m$ and $j = 2$, we get

$$0 \to \mathcal{O}(mD) \otimes \mathcal{I}_2 \to \mathcal{O}(mD) \to \mathcal{O}(mD)|_{D_2} \to 0$$

More subtly, if we take $l = m+1$ and $j = 1$; we get (since $\mathcal{O}((m+1)D) \otimes \mathcal{I}_1 = \mathcal{O}(mD) \otimes \mathcal{I}_2$),

$$0 \to \mathcal{O}(mD) \otimes \mathcal{I}_2 \to \mathcal{O}((m+1)D) \to \mathcal{O}((m+1)D)|_{D_1} \to 0$$

This shows that

$$\chi(X, \mathcal{O}((m+1)D)) - \chi(X, \mathcal{O}(mD)) = \chi(X, \mathcal{O}((m+1)D)|_{D_1}) - \chi(X, \mathcal{O}(mD)_{D_2})$$

Now D_j need not be irreducible or reduced, but one can again use filtering as above and induction to conclude that the RHS is a polynomial of order at most m^{n-1}. The claim of the Theorem now follows. □

5. Ample line bundles

Definition: A line bundle L on a projective variety X is said to be *very ample* if the complete linear system $H^0(X, L)$ is base-point free and the induced map $\phi_{|L|} : X \to \mathbb{P}(H^0(X, L))$ is a closed embedding. (This means that the map is injective, and a isomorphism of varieties onto the image Y; that is, $(\phi_{|L|})_* \mathcal{O}_X = \mathcal{O}_Y$. In other words, X is a subvariety of a projective space and $\mathcal{O}(1)$ restricts to L.) A line bundle L is *ample* if some positive power is very ample. A Cartier divisor D is ample (resp., very ample) if $\mathcal{O}(D)$ is ample (resp., very ample).

The basic result ("Cartan-Serre-Grothendieck" in Lazarsfeld) about ample line bundles is:

Theorem 5.1. *Let L be a line bundle on a projective variety X. Then following are equivalent:*

(1) L is ample.
(2) Given any coherent sheaf $\mathcal{F}$ on X, there is an integer $m_1(\mathcal{F})$ such that for $i > 0$,

$$H^i(X, \mathcal{F} \otimes L^m) = 0 \text{ provided } m \geq m_1(\mathcal{F}) \ .$$

(3) Given any coherent sheaf $\mathcal{F}$ on X, there is an integer $m_2(\mathcal{F})$ such that $\mathcal{F} \otimes L^m$ is globally generated for $m \geq m_2(\mathcal{F})$.
(4) There is a positive integer m_3 such that L^m is very ample for $m \geq m_3$.

Now, this is not the natural context for the result. One certainly needs compactness, which forces us to restrict to projective varieties. In the complex-analytic (Cartan's) context, the theorem would apply to compact complex manfolds; in the algebro-geometric context, the natural context is that of a possibly nonreduced and reducible abstract algebraic variety that is complete. For the definition of "completeness" see, eg., Mumford.

Before turning to the proof of the above Theorem, we recall some basics. Given a sheaf $\mathcal{F}$ on a topological space, denote by $\mathcal{F}_x$ its stalk at a point $x \in X$. By definition this is the direct limit

$$\lim_{\rightarrow} \Gamma(U, \mathcal{F})$$

where the limit is taken over open sets containing x, partially ordered by inclusion. If $\mathcal{F}$ is a coherent sheaf on a variety X,

$$\mathcal{F}_x = \Gamma(U, \mathcal{F}) \otimes_{\mathbb{C}[U]} \mathcal{O}_{U,x}$$

for any affine U containing x, where $\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$ is the local ring of X at x. The local ring $\mathcal{O}_{X,x}$ is itself the stalk of $\mathcal{O}_X$.

Let $\mathcal{I}(x) \subset \mathcal{O}_X$ denote the ideal sheaf of x, the sheaf of regular functions vanishing at x. (In the lectures I used the notation $\mathfrak{m}_{X,x}$, but this is better reserved for the maximal ideal in the local ring $\mathcal{O}_{X,x}$.) We have the defining exact sequence

$$0 \to \mathcal{I}(x) \to \mathcal{O}_X \to \mathbb{C}_{[x]} \to 0$$

where $\mathbb{C}_{[x]}$ is the skyscraper sheaf at x:

$$H^0(U, \mathbb{C}_{[x]}) = \mathbb{C} \text{ if } x \in U \text{ and } \{0\} \text{ otherwise}$$

and the map $\mathcal{O}_X \to \mathbb{C}_{[x]}$ is the evaluation map at z. (Thus $\mathbb{C}_{[x]} = \mathcal{O}_X/\mathcal{I}(x)$.) If $\mathcal{F}$ is a coherent sheaf, tensoring by it yields

$$\mathcal{I}(x) \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \to \mathbb{C}_{[x]} \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

(In general left-exactness is not preserved.) Denoting by $\mathcal{I}(x)\mathcal{F}$ the image of the sheaf morphism $\mathcal{I}(x) \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$, we get the exact sequence

$$0 \to \mathcal{I}(x)\mathcal{F} \to \mathcal{F} \to \mathbb{C}_{[x]} \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

Here $\mathbb{C}_{[x]} \otimes_{\mathcal{O}_X} \mathcal{F}$ is a skyscraper sheaf at x with "fibre" the finite-dimensional vector space $H^0(X, \mathbb{C}_{[x]} \otimes_{\mathcal{O}_X} \mathcal{F})$. (The term "fibre" is not standard except if $\mathcal{F}$ is locally free in which case this is indeed the fibre at x of the corresponding vector bundle[5].) A sequence of sheaves is exact iff the corresponding sequence of stalks is exact at every point, we have the exact sequence of modules over the local ring $\mathcal{O}_{X,x}$

$$0 \to (\mathcal{I}(x)\mathcal{F})_x \to \mathcal{F}_x \to H^0(\mathbb{C}_{[x]} \otimes_{\mathcal{O}_X} \mathcal{F}) \to 0$$

Let $\mathfrak{m}_x$ denote the maximal ideal in $\mathcal{O}_{X,x}$; then $(\mathcal{I}(x)\mathcal{F})_x = \mathfrak{m}_x \mathcal{F}_x$. We see that Nakayama's Lemma ("if M is a finitely generated module over a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}M = M$ iff $M = 0$") implies: *if $\mathcal{F}$ is coherent, then its fibre at x vanishes iff its stalk $\mathcal{F}_{X,x}$ vanishes, that is, iff $\mathcal{F}$ itself restricts to zero on some open neighbourhood of x.*

As a consequence:

Lemma 5.2. *Let $\mathcal{F} \to \mathcal{G}$ be a morphism of coherent sheaves on a variety X, and suppose that for some $x \in X$ the induced map (of finite-dimensional vector spaces)*

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathbb{C}_{[x]} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{C}_{[x]}$$

is onto. Then there is an open neighbourhood U of x on which the map of sheaves is onto.

[5]If E is a vector bundle, and $\underline{E}$ the corresponding sheaf, we reserve the right to use the notation E_x for the fibre and $\underline{E}_x$ for the stalk. In such contexts, we will stick to the notational distinction between E and $\underline{E}$

Proof. Let $\mathcal{K}$ denote the cokernel sheaf, so that we have the exact sequence:

$$\mathcal{F} \to \mathcal{G} \to \mathcal{K} \to 0$$

Tensoring with $\mathbb{C}_{[x]}$ we get the exact equence

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathbb{C}_{[x]} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathbb{C}_{[x]} \to \mathcal{K} \otimes_{\mathcal{O}_X} \mathbb{C}_{[x]} \to 0$$

which shows that $\mathcal{K} \otimes_{\mathcal{O}_X} \mathbb{C}_{[x]} = 0$. The previous Lemma applies and we see that $\mathcal{K}$ is zero on some neighbourhood of x. □

We will need another consequence of Nakayama's Lemma:

Lemma 5.3. *Let $f : (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ be a local homomorphism of Noetherian local rings, with A' finitely generated as an A-module and suppose that f induces an isomorphism of residue fields $A/\mathfrak{m} = A'/\mathfrak{m}' \equiv k$. For f to be surjective, it is (necessary and) sufficient that the induced map of "Zariski cotangent spaces"*

$$\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$$

is surjective.

Proof. We prove sufficiency. (My earlier proof had an error, as pointed out by Akashdeep; this one is adapted from Joe Harris: *Algebraic Geometry: A First Course.*) We will use the follwing corollary of Nakayama's Lemma repeatedly: *if A is a Noetherian local ring with maximal ideal $\mathfrak{m}$, M a finite A-module, and N a submodule such that $N + \mathfrak{m}M = M$, then $N = M$.*

By replacing A by its image in A', we can assume that the f is an injection. By assumption, the inclusion $\mathfrak{m} \to \mathfrak{m}'$ induces a surjection $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$, so

$$\mathfrak{m}A' + \mathfrak{m}'^2 = \mathfrak{m}'$$

Applying the above Corollary to the inclusion of A'-modules $\mathfrak{m}A' \subset \mathfrak{m}'$ yields $\mathfrak{m}A' = \mathfrak{m}$. Consider now the inclusion of A-modules $A \subset A'$. Since $\mathfrak{m}A' + A = \mathfrak{m}' + A = A'$, we see that $A = A'$. □

We can now turn to the proof of the Theorem.

Proof. (1) implies (2): Assume first that L is very ample. Then

$$X \underset{\iota}{\hookrightarrow} \mathbb{P}(V),$$

such that $\iota^*\mathcal{O}(1) = \underline{L}$. (We will need to talk about the fibre of L, so for a while we will keep the distinction between L and $\underline{L}$.) Then $H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^m) = H^i(\mathbb{P}(V), \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^m)) = H^i(\mathbb{P}(V), \iota_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}(V)}} \mathcal{O}(m))$. We have used the extension by zero operation ι_*; this takes a coherent sheaf on a closed subvariety to a coherent sheaf on the ambient variety. This is a special case of a direct image by a morphism. We have also used a special case of the projection formula: given a morphism $\pi : X \to Y$,

$$\pi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \pi^*\mathcal{V}) = \pi_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{V}$$

provided $\mathcal{V}$ is locally free[6]. Now, given any coherent sheaf $\mathcal{G}$ on $\mathbb{P}(V)$, there exists a $m(\mathcal{G})$ such that for $i > 0$,

$$H^i(\mathbb{P}(V), \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}(m)) = 0 \textit{ provided } m \geq m(\mathcal{G})$$

Applying this to $\iota_* \mathcal{F}$ gives the desired result, with $m_1(\mathcal{F}) = m(\iota_* \mathcal{F})$. If L is only ample, with L^M very ample, let ι be the corresponding projective embedding. We have, for $0 \leq l \leq M-1$ and $i > 0$,

$$H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^l \otimes_{\mathcal{O}_X} \underline{L}^{mM}) = 0 \textit{ provided } m \geq m(\iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^l)),$$

With a little book-keeping, this gives the desired result.

(2) implies (3): For $x \in X$, let $m_1(x)$ be such that for $i > 0$,

$$H^i(X, \mathcal{I}(x)\mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^m) = 0 \textit{ provided } m \geq m_1(x)$$

As a consequence, for $m \geq m_1(x)$, the evaluation map

$$H^0(\mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^m) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^m$$

is onto at x and therefore in an open neighbourhood U_x. By (quasi-)compactness we can cover X by finitely many of these open sets. Taking m_1 to be the supremum of the corresponding $m_1(x)$'s we are done.

(3) implies (4): For $x \in X$, consider the sheaf $\mathcal{I}(x)\underline{L}$; by (3) there exists $m(x)$ such that $\mathcal{I}(x)\underline{L} \otimes_{\mathcal{O}_X} \underline{L}^m$ is globally generated for $m \geq m(x)$. By quasi-compactness we can $\tilde{m}_3$ such that for $m \geq \tilde{m}_3 + 1$, the evaluation map

$$H^0(\mathcal{I}(x)\underline{L}^m) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{I}(x)\underline{L}^m \tag{5}$$

is onto for every $x \in X$. (Since $\underline{L}$ is invertible, $\mathcal{I}(x)\underline{L} \otimes_{\mathcal{O}_X} \underline{L}^l = \mathcal{I}(x)\underline{L}^{l+1}$.) Now $\mathcal{I}(x)\underline{L}^m$ is the sheaf of sections of $\underline{L}^m$ vanishing at x. So, given distinct $x, x' \in X$, there is a section of $\underline{L}^m$ that is nonvanishing at x' and vanishing at x. In other words, the linear system of $\underline{L}^m$ is base-point free and the corresponding map $\phi_{|L^m|} : X \to \mathbb{P}(H^0(X, \underline{L}^m))$ is injective. Since X is compact, the image is closed.

All that is left is to make sure that every (local) regular function on X is the pull back of a (local) regular function. Explicitly, let $f \in \mathcal{O}_{X,x}$; this is a regular function on $U \cap X$, with U open in $\mathbb{P}(V)$. We need to exhibit a $\tilde{f}$ regular on U, possibly at the cost of shrinking U. In other words, we have to ensure that for every $x \in X$, the map of local rings $\mathcal{O}_{\mathbb{P}(V),x} \to \mathcal{O}_{X,x}$ is onto. In turn, it suffices to show (Lemma 5.3) that $\mathfrak{m}_{\mathbb{P}(V),x}/\mathfrak{m}^2_{\mathbb{P}(V),x} \to \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ is onto. Consider the map

$$H^0(X, \underline{L}^m) \otimes_{\mathcal{O}_X} \underline{L}^{-m} \to \mathcal{O} \ .$$

This restricts to

$$H^0(X, \mathcal{I}(x)\underline{L}^m) \otimes_{\mathcal{O}_X} \underline{L}^{-m} \to \mathcal{I}(x) \ .$$

[6] As Pramath pointed out, if π is an affine morphism (as ι is) this holds even if $\mathcal{V}$ is not locally free. Proof: given a ring homomorphism $A \to B$, an A-module M_A and a B-module M_B, we have an isomorphism of A-modules $M_B \otimes_B B \otimes_A M_A \to M_B \otimes_A M_A$.

That this is onto for $m \geq m_3 \equiv \tilde{m}_3 + 1$ can be seen by tensoring (5) by $\underline{L}^{-m}$. This induces a surjective map of vector spaces

$$H^0(X, \mathcal{I}(x)\underline{L}^m) \otimes_{\mathbb{C}} L_x^{-m} \to \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$$

where L_x is a one-dimensional vector space, namely, the fibre of the line bundle L at x. For $y \in \mathbb{P}(V)$, we also have the analogous map

$$H^0(\mathbb{P}(V), \mathcal{I}_{\mathbb{P}(V)}(y)\mathcal{O}(1)) \otimes_{\mathbb{C}} \mathcal{O}(-1)_x \to \mathfrak{m}_{\mathbb{P}(V),y}/\mathfrak{m}^2_{\mathbb{P}(V),y}$$

which is also onto, although we seem not to need this (see below). But we have a commutative diagram:

$$\begin{array}{ccc} H^0(\mathbb{P}(V), \mathcal{I}_{\mathbb{P}(V)}(x)\mathcal{O}(-1)) \otimes_{\mathbb{C}} \mathcal{O}(-1)_x & \longrightarrow & \mathfrak{m}_{\mathbb{P}(V),x}/\mathfrak{m}^2_{\mathbb{P}(V),y} \\ {\scriptstyle =}\big\downarrow & & \big\downarrow \\ H^0(X, \mathcal{I}(x)\underline{L}^m) \otimes_{\mathbb{C}} L_x^{-m} & \xrightarrow{onto} & \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \end{array}$$

which shows that the map $\mathfrak{m}_{\mathbb{P}(V),x}/\mathfrak{m}^2_{\mathbb{P}(V),x} \to \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ is onto, provided we justify the equality $H^0(\mathbb{P}(V), \mathcal{I}_{\mathbb{P}(V)}(x)\mathcal{O}(1)) = H^0(X, \mathcal{I}(x)\underline{L}^m)$. To see this, consider the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^0(\mathbb{P}(V), \mathcal{I}_{\mathbb{P}(V)}(x)\mathcal{O}(1)) & \longrightarrow & H^0(\mathbb{P}(V), \mathcal{O}(1)) \\ & & \big\downarrow & & {\scriptstyle =}\big\downarrow \\ 0 & \longrightarrow & H^0(X, \mathcal{I}(x)\underline{L}^m) & \longrightarrow & H^0(X, \underline{L}^m) \end{array}$$

Finally, (4) implies (1) by definition.

□

We will join Lazarsfeld's losing battle and refer to the property of ampleness as "amplitude".

5.1. **Basic properties of amplitude.** (I) *If L is ample and L' is any line bundle $L' \otimes L^m$ is very ample for $m \geq m(L')$ large enough.* Proof: Let m_1 be such that L^m is very ample for $m \geq m_1$, and let m_2 be such that $L' \otimes L^m$ is globally generated for $m \geq m_2$. Then $L' \otimes L^m$ is very ample for $m \geq m_1 + m_2$

(II) *If L and M are ample line bundles on projective varieties X and Y respectively, $L \boxtimes M$ is ample on $X \times Y$.* Proof: We can suppose that L and M are both *very* ample. By Kunneth, $H^0(X \times Y, L \boxtimes M) = H^0(X, L) \otimes_{\mathbb{C}} H^0(Y, M)$, so $L \boxtimes M$ is free. The corresponding map $\phi_{|L \boxtimes M|}$ is the composition

$$\begin{aligned} X \times Y \underset{\phi_{|L|} \times \phi_{|M|}}{\to} & \mathbb{P}(H^0(X, L)) \times \mathbb{P}(H^0(Y, M)) \\ & \underset{V}{\to} \mathbb{P}(H^0(X, L) \otimes_{\mathbb{C}} H^0(Y, M)) = \mathbb{P}(H^0(X \times Y, L \boxtimes M)) \end{aligned}$$

where V is the "Veronese" embedding[7]. This shows that $L \boxtimes M$ is very ample.

(III) *If $f : Y \to X$ is a finite map of projective varieties and L is an ample line bundle on X, then f^*L is ample on Y.* Proof: Let $\mathcal{F}$ be a coherent sheaf on Y. We have for any m and i

$$\begin{aligned} H^i(\mathcal{F} \otimes f^*L^m) &= H^i(f_*(\mathcal{F} \otimes f^*L^m)) \\ &= H^i(f_*\mathcal{F} \otimes L^m)) \end{aligned}$$

where the first equality holds because f is affine, and the second is the projection formula. Now use part (2) of the Theorem. (Note that f^*L is trivial when restricted to any fibre of f. So if f is not finite, which in our context means if f has a positive-dimensional fibre, then f^*L is *not* ample.)

(IV) *If L is free, then it is ample iff the corresponding map $\phi_{|L|} : X \to \mathbb{P}(H^0(X, L))$ is finite iff deg $L|_C > 0$ for every irreducible curve C in X.*

(V) *Asymptotic Riemann-Roch II: Let X be an n-dimensional projective variety, $\mathcal{F}$ a coherent sheaf, D an* ***ample*** *Cartier divisor, $L = \mathcal{O}(D)$. Then there is a polynomial with rational coefficients which gives at integral values the function $m \mapsto \chi(\mathcal{F} \otimes L^m)$, and*

$$h^0(\mathcal{F} \otimes L^m) = (rank\ \mathcal{F})\frac{D^n}{n!}m^n + \text{ terms of order } m^{n-1} \text{ or lower}$$

where $h^i \equiv$ dim H^i. Proof: Since L is ample, $h^0(\mathcal{F} \otimes L^m) = \chi(\mathcal{F} \otimes L^m)$ for m large enough.

(VI) *Amplitude in families: If $f : X \to T$ is a proper morphism of quasiprojective varieties, and L a line bundle on X. For $t \in T$, set $X_t = f^{-1}(t)$. If $L|_{X_{t_0}}$ is ample for some $t_0 \in T$, then there is a neighbourhood U of t_0 such that $L|_{X_{t)}}$ is ample for $t \in U$.*

5.2. **Projective schemes.** For the most part we want to stick to varieties, but we will have to deal with more general schemes occasionally. Instead of biting the bullet and defining amplitude on schemes, we will manage with the following

Lemma 5.4. *Let $Y \subset \mathbb{P}(V)$ be a subscheme, $\mathcal{F}$ a coherent sheaf on Y, and L an invertible sheaf on Y such that $L|_{Y_1}$ is ample for every reduced, irreducible subscheme $Y_1 \subset Y$. Then for large enough m we have*

$$H^i(Y, \mathcal{F} \otimes_{\mathcal{O}_X} \underline{L}^m) = 0, i > 0\ .$$

Proof. We use the fact that $\mathcal{F}$ has a filtration

$$\mathcal{F} = \mathcal{F}_N \supset \mathcal{F}_{N-1} \supset \cdots \supset \mathcal{F}_0 = \{0\}$$

[7]Given finite-dimensional vector spaces V, W, the bilinear map $(\check{v}, \check{w}) \mapsto \check{v} \otimes \check{w}$ induces an embedding $\mathbb{P}_{sub}(\check{V}) \times \mathbb{P}_{sub}(\check{W}) \to \mathbb{P}_{sub}(\check{V} \otimes \check{W})$, and dually $\mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$.

which gives us exact sequences:

$$0 \to \mathcal{F}_{l-1} \to \mathcal{F}_l \to \mathcal{F}_l/\mathcal{F}_{l-1} \to 0$$

with successive quotients $\mathcal{F}_l/\mathcal{F}_{l-1}$ torsion-free on reduced, irreducible subvarieties Y_l of Y,. By hypothesis L is ample on all these subvarieties. Tensor with L^m, and choose m large enough that for $i > 0$

$$H^i(Y_l, \mathcal{F}_l/\mathcal{F}_{l-1} \otimes_{\mathcal{O}_X} \underline{L}^m) = 0,\ l = \ldots N$$

An upward induction on l now yields the desired vanishing of $H^1(Y, \mathcal{F} \otimes_{\mathcal{O}_X} L^m)$. □

6. Numerical criteria for amplitude

We begin with the criterion of Nakai-Moishezon-Kleiman:

Theorem 6.1. *A line bundle L on a projective variety X is ample iff for every irreducible $Y \subset X$*

$$c_1(L)^{(dim\ Y)}[Y] > 0$$

Proof. We prove first that if L is ample, then the inequality holds. Then some positive power L^m is very ample, and since $c_1(L^m) = mc_1(L)$, we have $c_1(L^m)^{(dim\ Y)}[Y] > 0 \implies c_1(L)^{(dim\ Y)}[Y] > 0$. So we can assume that $Y \subset (X \subset) \mathbb{P}(V)$, and L is the hyperplane bundle $\mathcal{O}(1)$. Now, the Bertini Theorem assures us that *for a generic hyperplane H_1, the intersection $H_1 \cap Y$ is an irreducible variety with $(H_1 \cap Y)^{sing} \subset Y^{sing}$.* By induction, we get hyperplanes $H_1, H_2, \ldots, H_{(dim\ Y-1)}$ such that $C \equiv Y \cap H_1 \cap H_2 \cap \cdots \cap H_{(dim\ Y-1)}$ is a (reduced, irreducible) curve with $C^{sing} \subset Y^{sing}$. One can now find a hyperplane $H_{(dim\ Y)}$ which avoids the singularities of C and intersects it transversally at its regular points. Then using our intersection theory black box, we get

$$c_1(L)^{(dim\ Y)}[Y] = \#\{Y \cap H_1 \cap H_2 \cap \cdots \cap H_{(dim\ Y)}\} > 0$$

(This integer is, by definition, the *degree* of Y with respect to the given projective embedding.)

Suppose now that the inequality $c_1(L)^{(dim\ Y)}[Y] > 0$ holds for every subvariety $Y \subset X$. We will prove that L is ample, by induction on $n = dim\ X$.

Step 1: Write $L = L \otimes M \otimes M^{-1}$ with both $L \otimes M$ and M very ample. By Bertini we can assume that $L \otimes M = \mathcal{O}(D)$ and $M = \mathcal{O}(D')$ with D and D' reduced and irreducible divisors. We have an exact sequence $0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, and a similar one involving D'. These yield, for any integer m:

$$0 \to \underline{L}^{m+1} \otimes_{\mathcal{O}} \mathcal{O}(-D) \to \underline{L}^{m+1} \to \underline{L}^{m+1}|_D \to 0$$

and

$$0 \to \underline{L}^m \otimes_{\mathcal{O}} \mathcal{O}(-D') \to \underline{L}^m \to \underline{L}^m|_{D'} \to 0$$

Noting that $\underline{L}(-D) = \mathcal{O}(-D')$, we get

$$0 \to \underline{L}^m \otimes_{\mathcal{O}} \mathcal{O}(-D') \to \underline{L}^{m+1} \to \underline{L}^{m+1}|_D \to 0$$
$$0 \to \underline{L}^m \otimes_{\mathcal{O}} \mathcal{O}(-D') \to \underline{L}^m \to \underline{L}^m|_{D'} \to 0$$

By our inductive hypothesis, the restriction of L to D is ample, as is its restriction to D'. There exists a m_3 such that for $m \geq m_3$,

$$H^i(D, L^m) = 0, \;\; i > 0$$
$$H^i(D', L^m) = 0, \;\; i > 0$$

This implies that for $i \geq 2$

$$H^i(X, L^m) = H^i(X, \underline{L}^m \otimes_{\mathcal{O}} \mathcal{O}(-D')) = H^{i+1}(X, L^m)$$

which shows that the function $m \mapsto H^i(X, L^m)$ is eventually constant. On the other hand, the Asymptotic Riemann-Roch (I) asserts

$$\chi(X, L^m) = \frac{c_1(L)^n[X]}{n!} m^n + \text{ terms of order } m^{n-1} \text{ or lower}$$

where $n = dim\ X$. We have thus

$$\begin{aligned} h^0(X, L^m) = & \frac{c_1(L)^n[X]}{n!} m^n + h^1(X, L^m) \\ & + \text{alternating sum of } h^i, \; i \geq 2 \\ & + \text{terms of order } m^{n-1} \text{ or lower} \end{aligned}$$

Now h^1 is nonnegative, the h^i, $i \geq 2$ bounded, and $c_1(L)^n[X] > 0$ by assumption, so we see that L^m *is effective for large enough* m.

Step 2: We will show that some power of L is globally generated. By replacing L by a suitable power, we can suppose that it has a nonzero section σ, with $(\sigma) = D$. In general D will have many components (all with codimension one) with multiplicities, so it need not be a variety. Consider:

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}|_D \to 0$$

where $\mathcal{O}|_D$ is the structure sheaf of the scheme D, pushed forward to X. Tensoring by $\underline{L}^m$, we get

$$0 \to \underline{L}^m(-D) \to \underline{L}^m \to \underline{L}^m|_D \to 0$$

By the inductive hypothesis L is ample restricted to each (reduced, irreducible) component of D, so by Lemma 5.4 $H^1(X, \underline{L}^m|_D)$ vanishes for large enough m, so that in that case we have the exact sequence:

$$H^1(X, \underline{L}^{m-1}) \to H^1(X, \underline{L}^m) \to 0$$

This shows that the $h^1(X, \underline{L}^m)$ is eventually constant and the maps eventually isomorphisms. In turn this shows that the maps

$$H^0(X, \underline{L}^{m-1}) \to H^0(X, \underline{L}^m|_D)$$

are eventually surjective. Let D' be any one of the irreducible components of D^{red}., and let $\mathcal{I}_{D'}$ be the ideal sheaf of D' in D. We have

$$0 \to \underline{L}^m \otimes \mathcal{I}_{D'} \to \underline{L}^m \otimes \mathcal{O}_D \to \underline{L}^m|_{D'} \to 0$$

Again using Lemma 5.4, $H^1(\underline{L}^m \otimes \mathcal{I}_{D'})$ vanishes for large enough m, so that eventually

$$H^0(\underline{L}^m \otimes \mathcal{O}_D) \to H^0(\underline{L}^m \otimes \mathcal{O}_{D'})$$

is onto. By the inductive hypothesis, $\underline{L}|_{D'}$ is ample, so $\underline{L}^m_{D'}$ is eventually globally generated. Now we can finish the proof of global generation as follows:

(1) $\underline{L}$ is globally generated outside D (by the section σ), and hence so is any positive power, and
(2) for a large enough m, on each component D_l of D, $\underline{L}^m$ is globally generated and every section of $\underline{L}^m$ on D_l extends to X.

Step 3: Let m be large enough that $\underline{L}^m$ is globally generated, so that we have a morphism

$$\phi_{L^m} : X \to \mathbb{P}(H^0(X, \underline{L}^m))$$

Note that $\mathcal{O}(1)$ pulls back to L^m, so that L^m is trivial on the fibres. This ensures that the fibres are zero-dimensional, since else on a curve $C \in X$ mapping to a point, we would have

$$c_1(L)[C] = 0$$

Thus ϕ_{L^m} is finite. Consequently L^m is ample, and so also L. □

The Nakai criterion has important corollaries.

Corollary 6.2. *Amplitude is stable under numerical equivalence.* (This lets us talk of the *ample cone* in the Neron-Severi group.)

Proof. This is a consequence of the fact, important in itself, that intersection numbers are defined modulo numerical equivalence. In other words, if Y is a k-dimensional variety and $L_1, \dots, L_k$ line bundles, the intersection

$$c_1(L_1) \wedge \cdots \wedge c_1(L_k)[Y]$$

depends only on the numerical equivalence class of the L_j. To see this, suppose $L_j \sim_{ne} L'_j$ where $\sim_{ne}$ indicates numerical equivalence. Then

$$\begin{aligned} c_1(L_1) \wedge \cdots \wedge c_1(L_k)[Y] - c_1(L'_1) \wedge \cdots \wedge c_1(L'_k)[Y] \\ &= \{c_1(L_1) - c_1(L'_1)\} \wedge \cdots \wedge c_1(L_k)[Y] \\ &+ c_1(L'_1)\{c_1(L_2) - c_1(L'_2)\} \wedge \cdots \wedge c_1(L_k)[Y] \\ &+ c_1(L'_1) \wedge \cdots \wedge \{c_1(L_k) - c_1(L'_k)\}[Y] \end{aligned}$$

so it suffices to consider the case when $L_j = L'_j = \mathcal{O}(D_j), j \geq 2$. Then

$$\begin{aligned} c_1(L_1) \wedge \cdots \wedge c_1(L_k)[Y] - c_1(L'_1) \wedge \cdots \wedge c_1(L_k)[Y] \\ &= \{c_1(L_1) - c_1(L'_1)\}[D_2 \cap \ldots D_k] \\ &= c_1(L_1)[D_2 \cap \ldots D_k] - c_1(L'_1)[D_2 \cap \ldots D_k] \end{aligned}$$

Now L_1 and L'_1 being numerically equivalent, their intersections with the curve $D_2 \cap \ldots D_k$ are equal. □

Corollary 6.3. *Let $f : Y \to X$ be a finite map of projective varieties, and suppose L is a line bundle on X such that f^*L is ample on Y. Then L is ample on X.*

Proof. Let $Y \subset X$ be an irreducible variety of dimension k, and let W be an irreducible component of the inverse image that maps onto Y. Then $c_1(f^*L)^k[W] = (degree\ f|_W) \times c_1(L)^k[Y] > 0$. □

7. Nef line bundles; Theorem of Kleiman

Consider the Neron-Severi group $N^1(X)$ of line bundles modulo numerical equivalence. This is finitely generated free abelian group of rank $\rho(X)$. Tensoring with $\mathbb{Q}$ (resp., $\mathbb{R}$) yields rational (resp., real) vector spaces of dimension $\rho(X)$, which we will denote $N^1(X)_{\mathbb{Q}}$ (resp., $N^1(X)_{\mathbb{R}}$).

(In case X is a smooth projective variety, the Lefshetz Theorem on $(1,1)$-classes identifies $N^1(X)_{\mathbb{Q}}$ with $H^2(X, \mathbb{Q}) \cap H^{(1,1)}(X, \mathbb{C})$.)

Recall that amplitude is preserved under numerical equivalence. Consider the subset $Amp(X)_{\mathbb{Z}}$ (this notation is not standard) of classes in $N^1(X)$ that correspond to ample line bundles. This is closed under addition because a tensor product of ample line bundles is ample. (It fails to be a semigroup because the trivial line bundle is not ample.)

Definition 7.1. A line bundle L on a projective variety X is *numerically effective* (*nef* for short) if $deg\ L|_C \geq 0$ for every (reduced, irreducible) curve $C \subset X$.

The subset $Nef(X)_{\mathbb{Z}}$ (this notation is not standard either) of nef classes in $N^1(X)$ is clearly a semigroup.

By a *cone* in a real vector space, we will mean a subset closed under multiplication by strictly positive scalars. We will now define two convex cones in the vector space $N^1(X)_{\mathbb{R}}$, the *ample cone* $Amp(X)$ and the *nef cone* $Nef(X)$. The ample cone is the convex cone generated by ample classes. Explicitly,

$$Amp(X) = \{\sum_j b_j[L_j] | b_j \geq 0,\ \sum_j b_j > 0,\ L_j\ ample\}$$

Note that if C is a (reduced, irreducible) curve, $[L] \mapsto c_1(L)[C]$ is a homomorphism $N^1(X) \to \mathbb{Z}$; it extends to a linear map $N^1(X)_{\mathbb{R}} \to \mathbb{R}$. The nef cone is defined by[8]

$$Nef(X) = \{\sum_j b_j[L_j] | \sum_j b_j c_1(L_j)[C] \geq 0 \ \forall\ C\}$$

We claim that

(1) $\emptyset \neq Amp(X) \subset Nef(X)$
(2) $Amp(X)$ is open, and
(3) $Nef(X)$ is closed.

In fact, we will see later that *the nef cone is the closure of the ample cone and the ample cone is the interior of the nef cone.*

Let us pause to justify the above claims. Since X is projective, the pull-back of the hyperplane bundle by any projective embedding is very ample, so ample classes exist. An ample class restricted to any curve C is ample and hence has positive degree. Thus

$$\emptyset \neq Amp(X)_{\mathbb{Z}} \subset Nef(X)_{\mathbb{Z}}$$

and (1) follows. That $Nef(X)$ is closed is clear because it is the intersection of closed "half-spaces". It remains to show:

Proposition 7.2. *$Amp(X)$ is open.*

Proof. Before going further, we fix a basis of generators $\{[L_i]|i = 1, \ldots, \rho(X)\}$ for the abelian group $N^1(X)$. This defines an L^∞-norm on the vector space $N^1(X)_{\mathbb{R}}$ which we will use to topologise it from now on.

We prove first: *let $[L]$ be an integral ample class. Then there exists $\epsilon(L) > 0$ such that every rational class $[L']$ with $||[L'] - [L]|| < \epsilon(L)$ is ample.* In fact, there exists $m_0 > 0$ such that $L^m L_i$ is ample for all i and $m \geq m_0$. Let $[L'] = [L] + \sum_i a_i[L_i]$, with a_i rational numbers. Then

$$[L'] = \frac{1}{\rho(X)} \sum_i \{[L] + a_i \rho(X)[L_i]\}$$

is ample provided $|a_i\rho(X)| \leq \frac{1}{m_0}$, i.e., $||[L'] - [L]|| < \epsilon(L) \equiv \frac{1}{\rho(X)m_0}$.

We prove next: *let $[L], [M]$ be integral classes, with L ample. Suppose give $\delta > 0$ such that $[L] + d[M]$ is ample for every rational d such that $|d| < \delta$. Then $[L] + dM]$ is ample for every real d such that $|d| < \delta$.* Proof: Pick rational numbers d_1, d_2 such that $-\delta < d_2 < d < d_1 < \delta$. Then

$$[L] + t[M] = \frac{d - d_2}{d_1 - d_2}([L] + d_1[M]) + \frac{d_1 - d}{d_1 - d_2}([L] + d_2[M])$$

which expresses $[L] + d[M]$ as a convex (real) linear combination of ample rational classes.

[8]Note that $\{\sum_j b_j[L_j]|b_j \geq 0,\ \sum_j b_j > 0,\ L_j\ nef\} \subset Nef(X)$, but apparently this can be a strict inclusion.

Finally suppose $[L] = \sum_j t_j[H_j]$ with t_j real and positive and $[H_j]$ integral and ample. Let $0 < t < t_1$ be rational. Then

$$[L] + \sum_i a_i[L_i] = t[H_1] + \sum_i a_i[L_i] + (t_1 - t)[H_1] + \sum_{j>1} t_j[H_j]$$

It is enough to show that $t[H_1] + \sum_i a_i[L_i]$ is ample for $|a_i|$ small enough. By the preceding paragraphs, there exists $\epsilon_1 > 0$ such that for each i, $[H_1]+b_i[L_i]$ for b_i real and $|b_i| < \epsilon_1$. Now note that

$$t[H_1] + \sum_i a_i[L_i] = \frac{t}{\rho(X)} \sum_i \{[H_1] + a_i \frac{\rho(X)}{t}[L_i]\}$$

and $[H_1] + a_i \frac{\rho(X)}{t}[L_i]$ is ample for $|a_i| < \frac{t}{\rho(X)}\epsilon_1$. □

We will prove below that $Amp(X)$ is the interior of $Nef(X)$ and $Nef(X)$ the closure of $Amp(X)$. This will be one of the corollaries of

Theorem 7.3. *(Kleiman) Let L ne a nef line bundle on a projective variety X. Then*

$$c_1(L)^{dim\ V}[V] \geq 0$$

for any reduced, irreducible subvariety $V \subset X$.

Proof. The proof will be by induction on $n = dim\ X$. The claim is true for $n = 1$. By induction, it suffices to prove the above inequality for $V = X$. We will suppose that the equality fails. i.e., that

$$c_1(L)^n[X] < 0$$

and derive a contradiction.

Let M be a very ample line bundle, and consider, for $t \in \mathbb{R}$,

$$P(t) = (c_1(L) + tc_1(M))^n[X]$$

Expanding the expression on the right we get

$$P(t) = c_1(L)^n[X] + tc_1(L)^{n-1}c_1(M)[X] + \cdots + t^n c_1(M)^n[X]$$

The coefficient of the t^{n-k} term is $c_1(L)^k c_1(M)^{n-k}[X]$; provided $k < n$, we can (by Bertini) compute this by taking $n-k$ generic hyperplane sections $H_1, \ldots, H_{n-k}$ such that their intersection is a reduced, irreducible variety of dimension k and evaluating $c_1(L)^k$ on this intersection:

$$c_1(L)^k c_1(M)^{n-k}[X] = c_1(L)^k[H_1 \cap \cdots \cap H_{n-k}]$$

By our inductive hypothesis these coefficients are all non-negative, and the coefficient of t^n (when $k = 0$) is positive (being the degree of X w.r.to the embedding given by M). If the constant term is negative, the polynomial P will have precisely one real root $t_0 > 0$.

Now write $P(t) = Q(t) + R(t)$ where

$$Q(t) = c_1(L)(c_1(L) + tc_1(M))^{n-1}[X]$$

and

$$R(t) = tc_1(M)(c_1(L) + tc_1(M))^{n-1}[X]$$

Now $R(t)$ is a polynomial with no constant term, and non-negative coefficients, and with the coefficient of the top degree term strictly positive. So $R(t_0) > 0$.

We will now show next that $Q(t_0) \geq 0$, which yields the desired contradiction since $Q(t_0) + R(t_0) = P(t_0) = 0$.

To prove that $Q(t_0) \geq 0$ it suffices to show that $Q(t) \geq 0$ for any rational number $\frac{a}{b} \equiv t > t_0$ (where a, b are coprime positive integers, of course). Note that

$$c_1(L) + tc_1(M) = \frac{1}{b}\{bc_1(L) + ac_1(M)\} = \frac{1}{b}c_1(L^bM^a)$$

so that Now given any k-dimensional subvariety $V \subset X$, with $k < n$,

$$c_1(L^bM^a)^k[V] = (bc_1(L)+ac_1(M))^k[V] = a^kc_1(M)^k[V]+\mathrm{non-negative terms} > 0$$

since M is ample and therefore the first term is positive. On the other hand,

$$c_1(L^bM^a)^n[X] = (bc_1(L) + ac_1(M))^n[X] = b^kP(\frac{a}{b}) > 0$$

Hence, by Nakai's criterion, L^bM^a is ample. Then

$$Q(t) = (\frac{1}{b})^{n-1}c_1(L)c_1(L^bM^a)^{n-1}[X] \geq 0$$

because L is nef. □

8. Consequences of Kleiman's Theorem

8.1. **Multiplicity.** We will need the notion of the multiplicity of a (reduced, irreducible) curve C at a point $x \in C$. For these matters, Fulton's book(s) on intersection theory are the standard reference. I also found Mumford's *Algebraic Geometry I* useful, as also *3264 & All That Intersection Theory in Algebraic Geometry.*

Let X be a variety and x a (closed) point. Let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ the maximal ideal. Consider the graded ring

$$\mathbb{C}[CT_xX] = \underset{\mathcal{O}_x/\mathfrak{m}_x}{\underset{=}{\mathbb{C}}} \oplus \mathfrak{m}_x/\mathfrak{m}_x^2 \oplus \cdots \oplus \mathfrak{m}_x^m/\mathfrak{m}_x^{m+1}$$

This is generated by $\mathfrak{m}_x/\mathfrak{m}_x^2$ and therefore defines a cone in the Zariski tangent space $T_xX \equiv Hom_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}^2)$. This affine scheme ("tangent cone") need not be reduced or irreducible (even if V itself is), but its (Krull) dimension is equal to the dimension of X at x. In the projective space $\mathbb{P}(\mathfrak{m}_x/\mathfrak{m}_x^2)$ this defines a sub-scheme ("projectivised tangent cone") of dimension $dim\ X-1$. The multiplicity $mult_x(X)$ of X at x is defined to be its degree:

$$mult_x(X) = c_1(\mathcal{O}(1))^{dim_x\ X-1}[\text{projectivised tangent cone}]$$

Even if X itself is reduced and irreducible, the tangent cone is likely to be neither, so our definition of intersection numbers has to be suitably extended.

If $X \subset \mathbb{C}^N$ is a hypersurface, then $mult_x(X)$ is characterised by

$$(\partial/\partial z_1)^{\alpha_1} \dots (\partial/\partial z_N)^{\alpha_N} f(x) = 0 \text{ iff} \sum_i \alpha_i < mult_x(X)$$

In other words $mult_x(X)$ is the degree of the lowest degree term in the Taylor expansion at x of a defining equation for X.

We will need three facts about multiplicities:

(1) If x is a regular point, the multiplicity is one. This is because $CT_xX = T_xX$.
(2) If x is a regular point of X, and C is a (reduced, irreducible) curve passing through x,

$$mult_x(C) = deg\ \mathcal{O}(E)|_{C'}$$

where E is the exceptional divisor of the blow-up $\mu : Bl_x(X) \to X$ of X at x and C' is the proper transform of C, i.e., the closure in $Bl_x(X)$ of $\mu^{-1}(C \setminus \{x\})$.
(3) if an effective divisor C and a curve D intersect at a singular point of either C or D, the "'intersection multiplicity" is bounded by the product of the multiplicities of C and D at that point. (See below.)

8.2. **Intersection Multiplicity.** Let X be projective variety $C \subset X$ an (irreducible, reduced) curve, and L a line bundle. According to our intersection theory black box,

(1) if σ is a section of L such that $(\sigma) = D = \sum_i D_i$, with each D_i an irreducible Weil divisor,
(2) C is not contained in any of the D_i,
(3) C is disjoint from the intersections $D_i \cap D_j$, for $i \neq j$,
(4) $x \in C \cap D$ iff x is a regular point of X, D and C, and
(5) the intersections $C \cap D$ are transverse,

then $c_1(L)[C] = D.C = \sum_i \#\{C \cap D\}$. What happens when conditions (4) and (5) are dropped? To cut a long story short,

$$c_1(L)[C] = D.C = \sum_i \sum_{x \in C \cap D_i} I(C, D_i, x)$$

where the $I(C, D, x)$ are *intersection multiplicities* which we will not define. The key fact which we will use is:

$$I(C, D, x) \geq mult_x(C) mult_x(D)$$

8.3. **Seshadri's Criterion.**

Theorem 8.1. *Let X be a projective variety, and L a line bundle on X. Then L is ample iff there exists $\epsilon(L) > 0$ such that for every (reduced, irreducible) curve C and every point $x \in C$, we have*

$$deg\ L|_C \geq \epsilon(L)\ mult_x\ C \tag{6}$$

(In other words, the degree of L restricted to any curve C should be bounded below uniformly in terms of the "maximum singularity" of C.)

Proof. Suppose first that L is ample. Then some power L^m has a section such that $E \equiv (\sigma)$ such that $\sigma|_C \neq 0$ but $\sigma(x) = o$. Thus E is effective, passes through x, and "meets C properly" (i.e., $C \nsubseteq E$.) Then

$$m\ deg\ L|_M = \sum_{y \in C \cap E} i(E, C, y) \geq i(E, C, x) \geq mult_x C$$

Here $i(E, C, y)$ is the intersection multiplicity of E and C at y. So (6) holds with $\epsilon(L) = 1/m$.

Conversely, suppose (6) holds for some positive $\epsilon(L)$. By induction[9] on dimension and using Nakai's criterion, it suffices to show that

$$c_1(L)^n[X] > 0$$

where $n = dim\ X$. Fix a smooth point $x \in X$, and let $\mu : X' \to X$ be the blow-up of X at x; let E be the exceptional divisor. Note that E is isomorphic to $\mathbb{P}^{n-1}$ and $\mathcal{O}(-E)|_E$ is the hyperplane bundle.

We claim that $L^m(-E)$ is nef on X' provided $\epsilon(L) < 1/m$. Granting this, we have by Kleiman's Theorem:

$$0 < (mc_1(L) - c_1(\mathcal{O}(E)))^n[X'] = m^n c_1(L)^n[X] + (-1)^n c_1(\mathcal{O}(E)))^n[X'] = m^n c_1(L)^n[X] - 1$$

which yields the desired inequality.

Turning now to the claim, let C' be any (reduced, irreducible) curve in X'. We need to show that

$$deg\ \mu^* L|_{C'} \geq \frac{1}{m} deg\ \mathcal{O}(E)|_{C'}$$

If $C' \subset E$, then L is trivial on C' and $\mathcal{O}(E))$ has negative degree on C'. If $C' \nsubseteq E$, let $C = \mu(C')$, so that C' is the proper transform of C. Now

$$deg\ \mu^* L|_{C'} = deg\ L|_C$$

and

$$deg\ \mathcal{O}(E)|_{C'} = mult_x(C)$$

so that by hypothesis, the claim is proved. □

8.4. **Other consequences of Kleiman's Theorem.**

Proposition 8.2. *Let X be a projective variety. Let z be a nef class and w an ample class. Then $z + w$ is ample.*

Proof. Let $\epsilon > 0$ be such that $Amp(X)$ contains an open ϵ-ball around w. Let z_1 be a positive rational linear combination of (integral) nef classes such that $||z - z_1|| < \epsilon/2$. Let w_1 be a rational ample class such that $||w_1|| < \epsilon/2$. It suffices to prove that $z_1 + w_1$ is ample. To see this note that

$$z + w = z_1 + w_1 + \{w + (z - z_1 - w_1)\}$$

[9]This induction takes care of positivity on subvarieties. These subvarieties could be contained in the singular locus of X, in which case we need to consider what happens at singular points $x \in X$. Thanks to Pramath for spotting this. For positivity on X itself we get by by considering a regular point x.

and (since $||(z - z_1 - w_1)|| < \epsilon$) the term $\{w + (z - z_1 - w_1)\}$ is ample.

By multiplying by z_1 and w_1 by an integer, we can assume that z_1 and w_1 are integral classes, with w_1 being the class of a very ample line bundle L. Now we check ampiltude of $z_1 + w_1$ using the Nakai criterion. Let V be any irreducible subvariety; then

$$(w_1 + z_1)^k[V] = w_1^k[V] + \ rest$$

The first term is positive since w_1 is ample; each of the other terms is (up to a positive constant factor) of the form

$$z_1^{k-l}[H_1 \cap \dots H_l \cap V]$$

where $l < k$ and H_i are irreducible divisors such that $\mathcal{O}(H_i) = L$. Each of the above intersection numbers is non-negative by Kleiman's Theorem. □

Corollary 8.3. *Fix an ample class w in $N^1(X)_{\mathbb{R}}$. A class $z \in N^1(X)_{\mathbb{R}}$ is ample if for some $\epsilon > 0$ such that for any curve $C \subset X$,*

$$z[C] \geq \epsilon w[C]$$

i.e., if $z - \epsilon w$ is nef.

Proof. $z = z - \epsilon w + \epsilon w$. □

As another corollary, we get:

Theorem 8.4. *Let X be a projective variety. Then*

(1) $\overline{Amp(X)} = Nef(X)$.
(2) $Amp(X) = Nef(X)^{interior}$.

Proof. The statement (1) is immediate: let z be a nef class, w any ample class, and for $t > 0$ consider the family of ample classes $z + tw$; as $t \to 0$ this tends to z. As for (2), let z be a nef class such that an open open ϵ-ball around z consists of nef classes. We wish to show that z is ample. Let w be ample with $||w|| < \epsilon/2$ and write $= z - w + w$. □

8.5. **The cone of curves.** Define $N_1(X)_{\mathbb{R}}$ to be the quotient of the real vector space generated by reduced, irreducible curves in X modulo numerical equivalence:

$$N_1(X)_{\mathbb{R}} = \{\sum_j a_j C_j | \text{red., irred., curve } C_j \hookrightarrow X, \ a_j \text{ real}\}/\{\textit{num. equiv.}\}$$

By definition, there is a non-degenerate pairing

$$N_1(X)_{\mathbb{R}} \times N^1(X)_{\mathbb{R}} \to \mathbb{R}$$

induced by $(C, [L]) \mapsto deg\ L|_C$. In particular $N_1(X)_{\mathbb{R}}$ is finite-dimensional of dimension $\rho(X)$. The *cone of curves* $NE(X)$ is the convex cone

$$NE(X) = \{\sum_j a_j C_j | \text{red., irred., curve } C_j \hookrightarrow X, \ a_j \text{ real}, \geq 0\}/\{\textit{num. equiv.}\}$$

Its closure $\overline{NE}(X) \subset N_1(X)_{\mathbb{R}}$ is the *closed cone of curves*.

Clearly,

$$\begin{aligned} Nef(X) &= \{z \in N^1(X)_{\mathbb{R}} | z(C) \geq 0, \;\; \forall\, C \underset{curve}{\subset} X\} \\ &= \{z \in N^1(X)_{\mathbb{R}} | z(\alpha) \geq 0, \; \forall\, \alpha \in NE(X)\} \\ &= \{z \in N^1(X)_{\mathbb{R}} | z(\alpha) \geq 0, \; \forall\, \alpha \in \overline{NE}(X) \end{aligned}$$

The theory of dual cones yields

$$\overline{NE}(X) = \{\alpha \in N_1(X)_{\mathbb{R}} | z(\alpha) \geq 0 \; \forall \; nef \;\; z\}$$

We finish with Kleiman's criterion for amplitude *via* cones.

Proposition 8.5. *A class $w \in N^1(X)_{\mathbb{R}}$ is ample iff $w(\alpha) > 0$ for all* nonzero *classes in $\overline{NE}(X)$. That is, iff $w \neq 0$ and $\overline{NE}(X) \setminus 0$ is contained in the open half-space defined by w:*

$$\overline{NE}(X) \setminus 0 \subset \{\alpha | w(\alpha) > 0\} \subset N_1(X)_{\mathbb{R}}$$

Proof. If w is ample, clearly $w(\alpha) \geq 0$ for $\alpha \in \overline{NE}(X)$. Suppose $w(\alpha) = 0$ for some nonzero $\alpha \in \overline{NE}(X)$. Let $z \in N^1(X)_{\mathbb{R}}$ such that $z(\alpha) < 0$. Then $z + tw$ is ample for large t, so $z(\alpha) = \{z + tw\}(\alpha) \geq 0$, yielding a contradiction.

Conversely, suppose w is a class such that

$$\overline{NE}(X) \setminus 0 \subset \{\alpha | w(\alpha) > 0\}$$

Consider the function $\alpha \mapsto w(\alpha)$ on the closed compact set:

$$\overline{NE}(X) \cap \{\alpha | ||\alpha|| = 1\}$$

Let ϵ be the minimum value. By assumption $\epsilon > 0$, which yields, for any curve C

$$w(C) > \epsilon ||[C]||$$

where $[C]$ is the class of C in $N_1(X)_{\mathbb{R}}$. Suppose that the norm in question is

$$||\alpha|| = \sum_i |w_i(\alpha)|$$

where w_i is a basis of $N^1(X)$ consisting of integral ample classes. (Such a basis exists because of the open-ness of $Amp(X)$.) This yields

$$w(C) > \epsilon w_1(C)$$

Now we appeal to Corollary 8.3. □

8.6. **Cones: the case of a smooth projective surface.** Let X be a smooth projective surface. The map

$$\sum_i a_i C_i \to \mathcal{O}(\sum_i a_i C_i), \;\; a_i \;\, integers, \;\; C_i \;\, curves \;\, in \;\, X$$

from divisors to line bundles yields (because it is clearly surjective) an isomorphism $N_1(X)_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}}$. Consider the symmetric bilinear form $\mathcal{B} : N_1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}} \to \mathbb{Q}$:

$$\mathcal{B}([\sum_i a_i C_i], [\sum_j a'_j C'_j]) = \sum_{i,j} a_i a'_j C_i.C'_j = \sum_{i,j} a_i a'_j c_1(\mathcal{O}(C_i))[C'_j]$$

Let $\mathcal{Q}$ be the corresponding quadratic form, so that

$$\mathcal{Q}(z) = \mathcal{B}(z, z)$$

One of the cornerstones of the theory of surfaces is the *Hodge Index Theorem* which says that the signature of $\mathcal{Q}$ is $(1, \rho(X) - 1)$. More precisely, if $w \in N^1(X)$ is the class of any ample line bundle, then

(1) $\mathcal{Q}(w) = w.w > 0$ (this is clear) and
(2) $\mathcal{Q}$ is negative definite on the orthogonal complement of w.

We extend $\mathcal{Q}$ and $\mathcal{B}$ in an obvious way to $N^1(X)_{\mathbb{R}}$. We let $\tilde{\mathcal{C}}$ denote the open one:

$$\tilde{\mathcal{C}} = \{z \in N^1(X) | \mathcal{Q}(z, z) > 0\}$$

This has two components, one of which contains $Amp(X)$; we let $\mathcal{C}$ denote the closure of this component.

Consider now the cones:

$$\begin{array}{ccccc} & & & \nearrow^{\subset_a} & \overline{NE}(X) \\ Amp(X) & \xrightarrow{\subset} & Nef(X) & & \\ & & & \searrow^{\subset} & \mathcal{C} \end{array}$$

(That $Amp(X) \subset NE(X)$ is clear, and the other inclusions follow by taking closures.)

(1) When is $Nef(X) \subset_a \overline{NE}(X)$ *not* an equality? Clearly iff there is an real effective divisor $\sum_i a_i C_i$ such that $\sum_i a_i C_i.C < 0$ for some curve C. Since $C.C' \geq 0$ for distinct irreducible curves, this can happen iff $C = C_i$ for some i and $C^2 < 0$.
(2) Suppose there is no curve C with negative self-intersection. Then $Nef(X) = \overline{NE}(X) \subset \mathcal{C}$. Since the interior of $Nef(X)$ consists of ample classes, any curve with C with zero self-intersection yields a nonzero class $[C] \in \partial\overline{NE}(X) \cap \{z \in \mathcal{C} | z^2 = 0\}$. Conversely, if $\partial\overline{NE}(X) \cap \{z \in \mathcal{C} | z^2 = 0\}$ is not empty, there exists such a curve.
(3) Suppose there does exist an irreducible curve C with negative self-intersection. Let $C_{\geq 0} \subset N^(X)$ denote the closed half-space of classes whose intersection with C is non-negative. Claim: $\overline{NE}(X)$ *is the cone spanned by* $\overline{NE}(X) \cap C_{\geq 0}$ *and* C, *and the half-line spanned by*

C is an extremal ray in $\overline{NE}(X)$. Proof: Let $u \in \overline{NE}(X) \setminus C_{\geq 0}$, $u \neq 0$. Then u is a limit of cycles $\sum_i a_i C_i$, with $a_i > 0$ and such that $\sum_i a_i C_i.C < 0$. At least one of the curves C_i must equal C. Now induct.

8.7. **Ruled surfaces: preliminaries.** Let C be an irreducible smooth projective curve (eventually of genus $g > 1$), E a rank 2 vector bundle on C, and let $\pi : X = \mathbb{P}(E) \to C$ be the corresponding projective bundle. The bundle $\mathbb{P}(E)$ carries a line bundle $\mathcal{O}(1)$, which sits in the exact sequence

$$0 \to \pi^*(det\ E) \otimes \mathcal{O}(-1) \to \pi^* E \to \mathcal{O}(1) \to 0\ ,$$

Given $p \in C$, the fibre F_p of π is the projective space of the vector space E_p, and $\mathcal{O}(1)$ restricts to $\mathbb{P}(E_p)$ as the corresponding tautological quotient line bundle. A quotient:

$$E \to \ell \to 0$$

with ℓ a line bundle, determines a section $\sigma : C \to X$ and a unique isomorphism $\sigma^*\mathcal{O}(1) \to \ell$ such that the following diagram commutes:

$$\begin{array}{ccccc} \sigma^*\pi^*E & \longrightarrow & \sigma^*\mathcal{O}(1) & \longrightarrow & 0 \\ {\scriptstyle =}\downarrow & & \downarrow & & \\ E & \longrightarrow & \ell & \longrightarrow & 0 \end{array}$$

The exact sequence of bundles on X

$$0 \to \Theta_\pi \to \Theta_X \to \pi^*\Theta_C \to 0$$

where Θ denotes the tangent bundle and $\Theta_\pi = \mathcal{O}(2) \otimes \pi^*(det\ E)^{-1}$ is the tangent bundle along the fibres, yields a formula for the canonical bundle of X:

$$K_X = \pi^*\{K_C \otimes (det\ E)\} \otimes \mathcal{O}(-2)$$

Fix $p \in C$. Since the Picard group of F_p is generated by $\mathcal{O}(1)|_{F_p}$, the restriction map $Pic(X) \to Pic(F_p)$ is onto. On the other hand, any line bundle from C, pulled back to X, clearly restricts to the trivial line bundle on F_p. We will show that in fact we have an exact sequence of abelian groups

$$0 \to Pic(C) \to Pic(X) \to Pic(F_p) \to 0$$

with $\mathcal{O}(1)$ providing a splitting of the restriction map $Pic(X) \to Pic(F_p)$.

In the following discussion, we will make use of cohomology and base-change as beautifully explained in Section 5 of Mumford's book *Abelian Varieties*. Let L be any line bundle on X. Since π is locally (on C) a product, $\mathcal{O}_X$ is flat over $\mathcal{O}_C$. Since L is locally free on $\mathcal{O}_X$, the sheaf $\underline{L}$ is flat over $\mathcal{O}_C$, so the base-change machinery is in place. If $L|_{F_p} = \mathcal{O}(b)|_{F_p}$, then this must hold for *all* fibres (with the same value of b since the topological type of L cannot jump), and the map $p \mapsto h^i(L|_{F_p})$ is constant. By Corollary 2 of Mumford, the direct images $R^i\pi_*L$ are locally free and obey base-change. Namely,

$$(R^i\pi_*L)_p = H^i(L|_{F_p})$$

where $(R^i\pi_*L)_p$ is the fibre at p of the vector bundle $R^i\pi_*L$.

Armed with all this, consider a line bundle L on X which is trivial on F_p. Thus L has degree zero and is therefore trivial on each fibre (because the fibre is a genus zero curve). Since $h^0(L|_{F_p}) = 1$, the direct image π_*L is a line bundle. Exercise: show that the tautological map $\pi^*\pi_*L \to L$ is an isomorphism. This shows that the above sequence of Picard groups is exact in the middle.

Suppose now that a line bundle M is pulled up from C. Note that

$$H^0(C, \pi_*\pi^*M) = H^0(C, M \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_X) = H^0(C, M \otimes_{\mathcal{O}_C} \mathcal{O}_C) = H^0(C, M)$$

So if M pulled up to X becomes trivial, $H^0(C, M)$ is one-dimensional, as is $H^0(C, \check{M})$, where $\check{M}$ is the dual line bundle. This can happen only if M is itself trivial. We have therefore shown that the arrow $Pic(C) \to Pic(X)$ is injective.

We want to describe the Neron-Severi group of X. This will turn out to be *all of* the second integral second cohomology of X, so let us first deal with this.

For a moment suppose that C is an arbitrary variety, E a vector bundle on C with $rank\ E = r$, and $X = \mathbb{P}(E)$ the corresponding projective bundle. Since the integral cohomology of any fibre F_p is generated by $\xi \equiv c_1(\mathcal{O}(1))$, the Leray-Hirsch theorem applies, and the integral cohomology of X is generated by ξ over $H^*(C, \mathbb{Z})$. More precisely,

(1) $H^*(X, \mathbb{Z}) = H^*(C, \mathbb{Z}) \oplus H^*(C, \mathbb{Z})\xi \oplus H^*(C, \mathbb{Z})\xi^2 + \cdots + H^*(C, \mathbb{Z})\xi^{r-1}$,

(2) the class ξ obeys:

$$\xi^r - c_1(E)\xi^{r-1} + \cdots + (-1)^c_r(E) = 0$$

where $c_l(E) \in H^{2l}(X, \mathbb{Z})$ is the l^{th} Chern class of E. (Note that the equation is written in the ring $H^*(X, \mathbb{Z})$, so by $c_l(E)$ we mean $\pi^*c_1(E)$.)

Let us revert to the case at hand. *We will suppose henceforth that det* $E = \mathcal{O}_C$. Since C is a curve and $c_1(E) = 0$ by assumption, $\xi^2 = 0$. The group $H^2(C, \mathbb{Z})$ is generated by $c_1(\mathcal{O}_C(p))$ for p any point in C. Pulling back to X, this is $f \equiv c_1(\mathcal{O}_X(F_p))$. Thus we see that

$$H^2(X, \mathbb{Z}) = \mathbb{Z}f \oplus \mathbb{Z}\xi = H^2(X, \mathbb{Z})^{alg} = N^1(X)$$

with $f^2 = 0$, $\xi^2 = 0$, and $f.\xi = 1$.

8.8. **Ruled surfaces: the nef cone and the cone of curves.** Before proceeding we need to make a definition:

Definition 8.6. Given a vector bundle E of rank $r > 0$ and degree d its *slope* is the ratio

$$\mu(E) \equiv \frac{deg\ E}{rank\ E}$$

The bundle E said to be *semistable* if given any (nonzero) sub-bundle $E' \hookrightarrow E$, we have $\mu(E') \leq \mu(E)$, or equivalently given a quotient bundle $E \twoheadrightarrow E''$, we have $\mu(E'') \geq \mu(E)$. In particular if E is of rank 2 and degree zero, it is *non-semistable* if there exists a quotient line bundle $E \twoheadrightarrow L$, with $deg\ L = -d,\ d > 0$. (Exercise: In the latter case the bundle L is unique.)

Consider a class $af + b\xi \in N^1(X)_{\mathbb{R}}$. If this is nef, its restriction to any fibre must have non-negative degree, which forces $b \geq 0$. Note also that $(ad+b\xi)^2 = 2ab$, so the nef cone has to be contained in the positive quadrant $\mathcal{C} = \{a \geq 0, b \geq 0\}$.

Case 1: E non-semistable. Suppose now that E is non-semistable. The destabilising quotient L determines a section $\sigma : C \to X$ such that $\sigma^*\mathcal{O}(1) = L$. Let $\tilde{C} \subset X$ denote the image of C by σ. Since $deg\ \xi|_{\tilde{C}} = deg\ L|_C = -d$ and $\tilde{C}.F_p = 1$, we have $[\tilde{C}] = \xi - df$ in $N^1(X)$. In particular $[\tilde{C}]^2 = -2d$. (This can also be seen as follows. Note that $(\tilde{C})|_{\tilde{C}}$ is the normal bundle to $\tilde{C}$, which in turn identified with can be identified with $\Theta_\pi|\tilde{C} = \mathcal{O}(2)|\tilde{C}$, so $\tilde{C}^2 = deg\ \mathcal{O}(\tilde{C})|\tilde{C} = deg\ \mathcal{O}(2)|_{\tilde{C}} = deg\ \sigma^*(\mathcal{O}(2))|_C = 2deg\ L|_C = -2d.$)

If a class $af + b\xi$ is nef, its restriction to $\tilde{C}$ must have non-negative degree, which forces $a - bd \geq 0$, or equivalently,

$$b/a \leq 1/d$$

We claim that in fact

$$Nef(X) = \{a \geq 0, b \geq 0,\ and\ b/a \leq 1/d\}$$

from which we can conclude that

$$Amp(X) = \{a > 0, b > 0,\ and\ b/a < 1/d\}$$

To justify the claim, consider the cone of curves $\overline{NE}(X) \subset N^1(X)$. This contains the closed half-f-axis $\{af | a \geq 0\}$ as well as the half-line generated by the class $[\tilde{C}] = \xi - df$. Since $\tilde{C}$ has negative self-intersection, it spans an extremal ray in $\overline{NE}(X)$ which therefore is bounded by the half lines spanned by f and $\xi - df$. The dual cone is exactly $\{a \geq 0, b \geq 0,\ and\ b/a \leq 1/d\}$.

Case 2: E semistable. We will use the fact that, as a consequence[10] of the Theorem of Narasimhan and Seshadri, the symmetric powers S^mE are also semistable. (Exercise: show by induction that $det\ S^mE \sim \mathcal{O}_X$.) If $\tilde{C}$ is a (reduced irreducible) curve in X, and not a fibre F_p, then $\pi : \tilde{C} \to C$ is a finite surjective map of degree m (say). The curve $\tilde{C}$ is the divisor of zeros of a (nonzero) section of $\mathcal{O}(m) \otimes \pi^*A$ for some line bundle A on C. This can be viewed as a (nonzero) morphism from the dual line bundle: $\check{A} \to S^mE$. This induces an injective map of bundles $\check{A}(D) \to S^mE$, where D is some effective divisor on C, possibly trivial. By semistability of S^mE, we have

$$deg\ \check{A} + deg\ D \leq 0 = \frac{deg\ S^mE}{m+1}$$

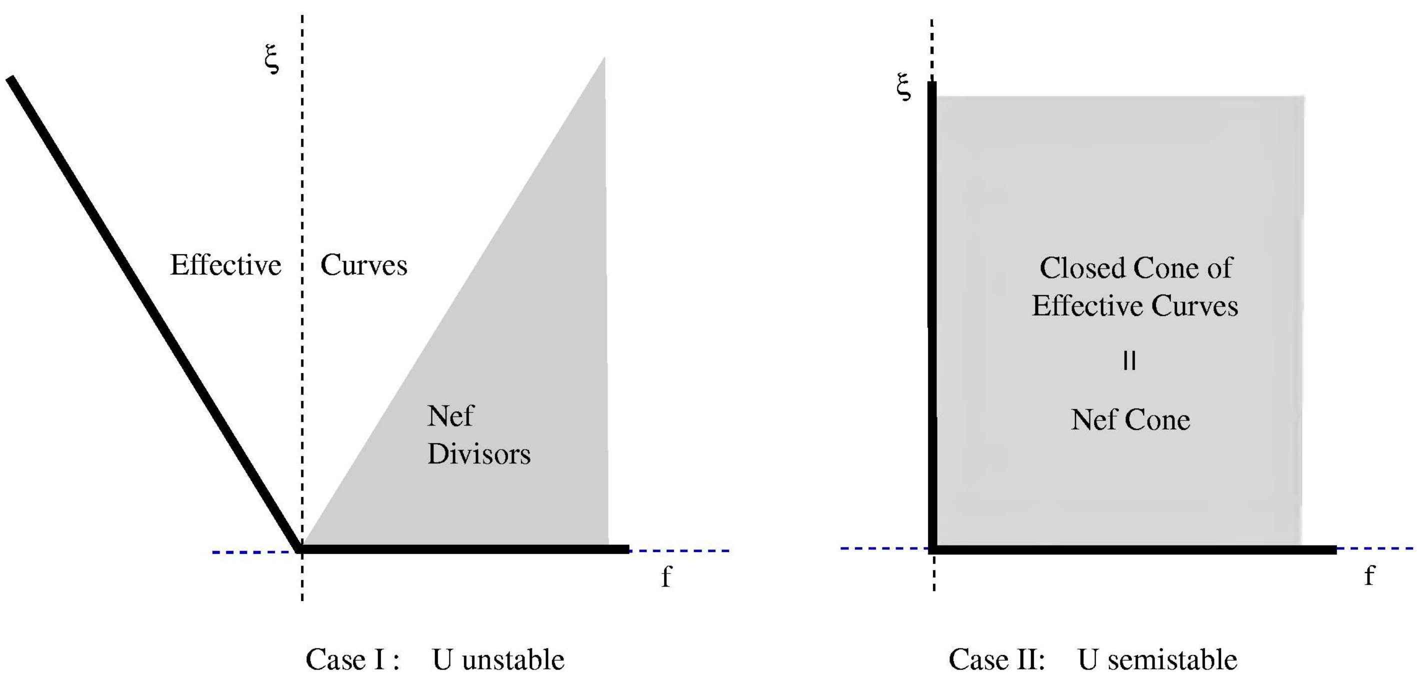
[10]There are also algebro-geometric proofs avoiding appeal to the Narasimhan-Seshadri Theorem.

In particular $deg\ \check{A} \leq 0$, or equivalently, $a \equiv deg\ A \geq 0$. Thus $\mathcal{O}(\tilde{C}) = af + m\xi$, with $m > 0, a \geq 0$, so we have proved that

$$NE(X) \subset \{af + b\xi | a \geq 0, m > 0\}$$

Since $Nef(X) \subset \overline{NE}(X)$ and these cones are mutually dual, it follows that $Nef(X) = \overline{NE}(X) = \{af + b\xi | a \geq 0, m \geq 0\}$.

Here is a very helpful figure copied and pasted from Lazarsfeld's book (His U is our E.):



8.9. **More on the cone of curves.** Can it happen that $NE(X)$ is not closed? As we saw above, the closure $\overline{NE}(X)$ contains the ξ-axis, but we will show that there is no curve C such that $[C]$ lies on this axis, provided E is suitably chosen. If this were the case, we would have

$$\mathcal{O}(C) = \mathcal{O}(m) \otimes \pi^* A$$

for some $m > 0$ and a line bundle of degree zero on C. As we argued above this implies that the bundle $S^m E$ contains the degree zero line sub-bundle $(\check{A})$. If we can find E such that $S^m E$ is stable for every $m > 0$ (and such bundles exist for *genus* $C > 1$), this cannot happen. The existence of such an E follows from the Narasimhan-Seshadri Theorem.

Here is a proof supplied by Narasimhan. It is a fact that there exist two elements A_1, B_1 in $SU(2)$ that generate a dense subgroup. (In fact this is true for $SU(N),\ N \geq 2$.) Let $A_2, B_2, \ldots, A_g, B_g$ be elements of $SU(2)$ defined by $A_2 = B_1, B_2 = A_1$ and $A_i = B_i = I,\ i > 2$. Then

$$\prod_i A_i B_i (A_i)^{-1} (B_i)^{-1} = I$$

so we have a representation of the fundamental group of C in $SU(2)$. Since the symmetric powers of the standard two-dimensional representation (on $\mathbb{C}^2$) of $SU(2)$ are irreducible representations of $SU2$), the induced representations of the fundamental group in $SU(m+1) = SU(S^m\mathbb{C}^2)$ are irreducible.

Continuing with the above example, if $X = \mathbb{P}(E)$ with E stable of degree zero, then $\mathcal{O}(1)$ has positive degree on every curve but is not ample. For, given any curve C we have

$$[C] = a\xi + bf$$

with $b > 0$. All this is mostly due to Mumford.