Theory of Computation
Homework 1

1. Design a DFA over the alphabet $\Sigma = \{0, 1\}$ that accepts the language $L = \{w \mid w \text{ is the binary representation of a number divisible by } 4\}$. Consider the empty string to be a representation of 0. Formally prove that the DFA you have designed accepts precisely the language specified.

2. There is a wolf, a goat, a cabbage, a boat and an oarsman on one side of a river. The goal is for everyone to cross over to the other side, but the boat can take only one passenger apart from the oarsman. A string over the alphabet $\Sigma = \{w, g, c, o\}$ can be interpreted as instructions to the oarsman as follows: if the oarsman is on some side, the letter w instructs the oarsman to take the wolf with him on the boat and cross over to the other side. The letters g and c are interpreted as similar instructions. The letter o instructs the oarsman to take the boat alone and cross the river. Design a NFA over Σ to accept the set of strings that satisfy the following conditions:

   (a) The string should not lead to an instruction impossible to perform: if at some stage, the oarsman is to take the wolf in the boat, the oarsman and the wolf should be one the same side. Similar conditions apply to the goat and the cabbage.

   (b) At no stage should the wolf and the goat be on the same side with oarsman on the other side.

   (c) At no stage should the goat and the cabbage be on the same side with oarsman on the other side.

   (d) Starting with all four on one side, the string should lead to all four being on the other side.

3. The upward closure of a string w is defined as the set $\text{uc}(w) = \{w' \mid w \text{ can be obtained by deleting some letters in } w'\}$. For example $aaabbabb$, $abbba \in \text{uc}(aba)$ but $aabbb \notin \text{uc}(aba)$. The upward closure of a language L is defined as $\text{uc}(L) = \bigcup_{w \in L} \text{uc}(w)$. Prove that if L is a regular language, then so is $\text{uc}(L)$.

4. Let a restricted DFA be any DFA that has only one accepting state. Are restricted DFAs strictly less powerful than DFAs? If yes, give an example of a language that is accepted by a DFA but that can not be accepted by any restricted DFA. If no, prove that any language accepted by a DFA can also be accepted by some restricted DFA.

5. For any language L, let $L_{-\frac{1}{3}-}$ be the language consisting of middle one-thirds of strings in L:

$$L_{-\frac{1}{3}-} = \{w \mid \text{for some } w_1, w_3, |w_1| = |w| = |w_3| \text{ and } w_1 \cdot w \cdot w_3 \in L\} \ .$$

Prove that if L is regular, so is $L_{-\frac{1}{3}-}$. Hint: run three copies of the automaton accepting L.

6. For any regular language L, prove that the language $\{w \mid ww^r \in L\}$ is regular, where w^r is the string w reversed.

7. For a NFA A, the language $\mathcal{L}(A)$ is defined as $\mathcal{L}(A) = \{w \mid A$ has a run on w starting from some initial state and ending at some final state$\}$. Consider the language $\mathcal{L}'(A) = \{w \mid A$ has a run on w starting from some initial state and ending at some final state; and any run of A on w starting from any initial state ends at some final state$\}$. Is it possible to design a DFA to accept the language $\mathcal{L}'(A)$? Design an algorithm to check if $\mathcal{L}(A) = \mathcal{L}'(A)$.

8. Let $\Sigma = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be the set of four ordered pairs made up of 0s and 1s. Given a string $w = \sigma_1 \sigma_2 \cdots \sigma_n$, we obtain two binary numerals w^t and w^b by reading the sequence of bits at the top of each σ_i and the sequence of bits at the bottom of each σ_i. For example, if $w = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $w^t = 0001$ and $w^b = 0101$. Design a DFA over the alphabet Σ that accepts the language $\{w \mid w^t + 1 = w^b\}$. Hint: regular languages are closed under reversal.