CMI/MSc Measure Theory Home Work 1 04-08-2010 .

If we are going to measure sets, we better know our sets and operations on sets well.

1. Show that the collection of all subsets of a non-empty set Ω with the binary operation Δ of symmetric difference forms a group. Recall that for two sets A and B, we define $A\Delta B = (A \cup B) - (A \cap B)$, that is, all points which are in exactly one of the two sets.

2. Let $\{A_n\}$ be a sequence of subsets of a given set Ω. Recall that lim inf of this sequence of sets is precisely the set of points that belong to ALL of these sets after some stage, equivalently, those points that fail to belong to finitely many of these sets. Similarly lim sup is the set of all points that belong to infinitely many of these sets. Show that

$$\limsup A_n = \cap_m \cup_{n>m} A_n \quad ; \quad \liminf A_n = \cup_m \cap_{n>m} A_n$$

   Recall the definitions of lim inf and lim sup for a sequence of real numbers (a_n). What is the analogue of the above display for numbers?

   Show that $\liminf A_n \subset \limsup A_n$. What is the analogue for numbers?

   State DeMorgans laws for lim sup and lim inf of sets. What is the analogue for numbers?

   You know lim inf and lim sup of functions defined point wise. What is the relation between lim sup/ lim inf of sets and lim sup/lim inf of their indicator functions?

   Suppose that we have a class $\mathcal{C}$ of subsets of Ω which is closed under countable unions and countable intersections. If all these sets A_n belong to $\mathcal{C}$, show that their lim inf and lim sup also belong to $\mathcal{C}$.

3. Here is a special subset of the Real line, called the Cantor middle third set. Let $I_0 = [0,1]$, $I_1 = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$, and so on. In general, if I_n is defined as the union of 2^n disjoint closed intervals each of length 3^{-n}, then I_{n+1} is the union of 2^{n+1} disjoint closed intervals obtained by deleting the middle open one-third interval from each of the 2^n intervals in I_n. Let $C = \bigcap I_n$.

   Show that the sum of the lengths of the deleted intervals adds to one. However C itself is nonempty. In fact show that C is an uncountable closed set.

   This set C is called a Cantor set. (Do not bother now about my using 'a' instead of 'the' Cantor set, but when we say Cantor set we mean only this set.)

   Show that every number in the unit interval $[0,1]$ has a ternary expansion. That is given $0 \leq x \leq 1$, we can get numbers $x_1, x_2, \cdots$ such that each x_i

is one of the numbers $0, 1, 2$ and

$$x = \frac{x_1}{3} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \cdots.$$

A number may have two such expansions. Show that a number x is in C iff it has a ternary expansion without using the digit one.

Here is a map from C to the unit interval. Take $x \in C$, say having ternary expansion

$$x = \frac{2y_1}{3} + \frac{2y_2}{3^2} + \frac{2y_3}{3^3} + \cdots.$$

wher each y_i is either zero or one, remember x has ternary expansion with digits zero and two only. Put

$$f(x) = \frac{y_1}{2} + \frac{y_2}{2^2} + \frac{y_3}{2^3} + \cdots.$$

Thus I change the ternary expansion to binary expansion. Show that f is a continuous function on C onto $[0, 1]$. Is it one-to-one? For a given $a \in [0, 1]$ show that $f^{-1}(a)$ has at most two points in C. Further the set of numbers a for which $f^{-1}(a)$ has two points forms a countable set.

4. Let us make a notation to save repeating long phrases. The collection of open subsets of the real line is denoted $\mathcal{G}$ and the collection of closed sets by $\mathcal{F}$. For *any* class of sets $\mathcal{C}$, let us denote by $\mathcal{C}_\sigma$ the collection of sets obtained by taking countable union of sets in the class $\mathcal{C}$ and let us denote by $\mathcal{C}_\delta$ the collection of sets obtained by taking countable intersections of sets in the class $\mathcal{C}$.

   For example $(0, 1)$ is in $\mathcal{G}$ but not in $\mathcal{F}$ where as $[0, 1]$ is in $\mathcal{F}$ but not in $\mathcal{G}$. Show this (easy).

   Since union of open sets is again open, observe $\mathcal{G}_\sigma$ is just the class $\mathcal{G}$ itself. Similarly the class $\mathcal{F}_\delta$ is the class $\mathcal{F}$ itself. You get nothing new.

   The set Ir of irrational numbers is in $\mathcal{G}_\delta$ but not in $\mathcal{F}_\sigma$ where as the set Q of rational numbers is in $\mathcal{F}_\sigma$ but not in $\mathcal{G}_\delta$. Show this (not so easy).

   If you take the set A to be the union of the two sets : irrationals in $[0, 1]$, rationals in $[10, 11]$. This set is in none of the above classes. Indeed this set is in $\mathcal{G}_{\delta\,\sigma}$. Show this.

5. Define a sequence of functions F_n on R as follows: Each of the functions equals 0 on $(-\infty, 0)$ and equals 1 on $(1, \infty)$. Here is the definition on $[0, 1]$. F_0 is the polygonal line joining the points $(0, 0)$ and $(1, 1)$. F_1 is the polygonal line joining the points $(0, 0), (\frac{1}{3}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2})$ and $(1, 1)$; in that order. F_2 is the polygonal line joining $(0, 0), (\frac{1}{9}, \frac{1}{4}), (\frac{2}{9}, \frac{1}{4}), (\frac{1}{3}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2}), (\frac{7}{9}, \frac{3}{4}), (\frac{8}{9}, \frac{3}{4})$ and $(1, 1)$; in that order. Thus F_n remains flat on each of the intervals deleted so far and in each one of the remaining 2^n intervals it is a straight

line such that its values at the two end points differ by $1/2^n$. You do not understand this unless you have drawn yourself the graphs of the functions F_1, F_2 and F_3. Show that the sequence F_n converges uniformly to a continuous function F on R and that F is a non-decreasing function.

Right now we keep this function in our kit and pull out later for use.

This function F is called the Cantor distribution function.

6. Suppose that $\mathcal{A}$ is a field of subsets of some set. Assume that there are only finitely many sets which belong to this field. How many sets could there be in the field? Can it have exactly 200 sets? Can it have exactly 1024 sets?

7. Consider $\Omega = \{0, 1, 2 \cdots\}$. For each $n \geq 0$, let a_n be a nonnegative number. Define for every subset $A \subset \Omega$, $\mu(A) = \sum_{n \in A} a_n$. Empty sum is, by definition, zero. Show the following:

   (i) μ is a nonnegative function defined for all subsets of Ω.

   (ii) $\mu(\emptyset) = 0$. $\mu(A) < \infty$ for every finite set A.

   (iii) $\mu(\cup A_i) = \sum \mu(A_i)$ for disjoint sequence of sets A_i.

   Conversely, suppose that μ is a function defined for all subsets of Ω satisfying the three conditions above. Show that there are indeed nonnegative numbers a_n and the given μ is obtained by the formula prescribed above.

   Is it necessary to have Ω to be the set of nonnegative integers in the above problem? What if it were some countable set?

8. Consider the set $2^\omega = \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \cdots$, the set of infinite sequences $(\epsilon_1, \epsilon_2, \cdots)$. of zeros and ones. If you take a finite sequence s of zeros and ones we can define $A_s \subset \Omega$ as the set of all infinite sequences which start with s, that is, those infinite sequences whose i-th coordinate agrees with the i-th coordinate of s for each i upto length of s; remember s is only a finite sequence. If you take s to be the empty sequence e, then its length is zero and hence there is no restriction and thus we get $A_e = \Omega$. Let $\mathcal{S}$ be the collection of sets A_s as s ranges over all finite (including empty) sequences of zeros and ones. Let us put emptyset also in this collection.

   Show that $\mathcal{S}$ is a semifield of subsets of Ω. State precisely and prove the following: If a set in $\mathcal{S}$ is a disjoint union of sets in $\mathcal{S}$, then actually it is a finite union.

   Do you see why we should be interested in this?

We shall not meet from Aug 13 to 29. Next class after Wednesday, 11 Aug will be Monday, Aug 30.

9. Let $\Omega = R$ and $\mathcal{F}$ be the collection of all sets $A \subset \Omega$ such that either A is finite or $\Omega - A$ is finite. Show that this is a field of subsets of Ω. Is it a σ-field? What if Ω were the set of integers instead of R? What if Ω were a finite set?

   Consider the above field. Define $P(A) = 0$ or 1 according as A is a finite set or not. Show that P is finitely additive probability. Show that it is not countably additive when Ω is a countably infinite set. Show also that it is countably additive if Ω is either finite or uncountable.

10. Let $\Omega = R$ and $\mathcal{A}$ be the collection of sets $A \subset R$ such that either A is countable or $\Omega - A$ is countable. Show that this is a σ-field of subsets of Ω. Define $P(A) = 0$ or 1 according as A is countable or not. Show that this is indeed a probability on $\mathcal{A}$.

11. It is interesting to note that the probability in the above example takes only two values zero and one, but yet the probability of each singleton set is zero. It is natural to ask if I should cook up such apparently useless σ-fields to exhibit $0 - 1$ valued probabilities. The answer is: yes!

    More precisely show the following : Let P be a probability on the Borel σ-field $\mathcal{B}$ of R. Suppose that P takes only two values 0 and 1. Then show that there is a number $a \in R$ such that $P\{a\} = 1$. In particular for any Borel set B, $P(B)$ is 1 or 0 according as the Borel set includes a or not.

    What is so special about R ? Nothing. We shall discover this slowly. First some practice with the Borel σ-field.

12. Consider $\Omega = R$. In the next sentence a and b are real numbers, not $\pm\infty$. Let $\mathcal{C}$ be any one of the following collections of subsets of R : all intervals $(-\infty, a)$ OR all intervals (a, ∞) OR all intervals $(-\infty, a]$ OR all intervals $[a, \infty)$ OR all intervals (a, b) OR all intervals $(a, b]$ OR all intervals $[a, b)$ OR all intervals $[a, b]$ OR the same classes with a and/or b restricted to rationals OR the collection of all open sets OR the collection of all closed sets. Show that the the Borel σ-field $\mathcal{B} = \sigma(\mathcal{C})$. What if I replaced the word rationals by D where D is a fixed countable dense subset of R.

13. You must be wondering that I am obsessed with R and σ-fields on R. Not really. Consider the cartesian plane R^2. Consider the class of sets $A \times B$ where A and B are Borel subsets of R. Denote this class by $\mathcal{R}$. These are called Borel rectangles. Denote $\sigma(\mathcal{R})$ by $\mathcal{B}^2$ OR sometimes by just $\mathcal{B}$. This is called the Borel σ-field on R^2.

    Show that the open unit disc belongs to $\mathcal{B}^2$. Show that every open subset of R^2 is in $\mathcal{B}^2$. Show that every closed set also belongs. In particular, the sets $\{(x, y) : x \leq y\}$; $\{(x, y) : x \geq y\}$; $\{(x, y) : x = y\}$; $\{(x, y) : 5x+21y = 93\}$ are all Borel subsets of R^2.

Fix any one of the classes $\mathcal{C}$ of the previous exercise. Let $\mathcal{R}^1$ be the collection of sets $A \times B$ where $A, B \in \mathcal{C}$. Just to get practice show that $\mathcal{B}^2 = \sigma(\mathcal{R}^1)$.

Instead of one of the above families, suppose that I fix two of the classes $\mathcal{C}$ and $\mathcal{D}$ of the previous exercise and considered $\mathcal{R}^1$ to be the collection of sets $A \times B$ where $A \in \mathcal{C}$ and $B \in \mathcal{D}$. We still have $\mathcal{B}^2 = \sigma(\mathcal{R}^1)$. Show this.

14. Say that a σ-field $\mathcal{A}$ is *countably generated* if there is a countable family $\mathcal{C}$ of sets such that $\mathcal{A} = \sigma(\mathcal{C})$.

    For instance the Borel σ-field on R is countably generated. The power set on integers is a countably generated σ-field. Do you think the countable - cocountable σ-field described earlier is countably generated? –no,not a tongue twister. The Borel σ-field on R^2 is also countably generated.

    Countably generated σ-fields are not difficult to obtain. You can start with your own choice of set Ω and with your own favourite countable collection $\mathcal{C}$ of subsets of Ω and just consider $\sigma(\mathcal{C})$.

    Suppose that $\mathcal{A}$ is a countably generated σ-field of subsets of Ω, generated by, say, $\mathcal{C} = \{C_1, C_2, \cdots\}$. For each i take C_i OR complement of C_i – you make your choice. Denote it by B_i. Let $A = \cap_i B_i$. Clearly $A \in \mathcal{A}$. Suppose that $A \neq \emptyset$. Then A is one of the smallest sets available in the σ-field. More precisely show that if $A_0 \in \mathcal{A}$ is a proper subset of A then A_0 must be the empty set. Such sets are called *atoms* of the σ-field.

    To continue the story, suppose now that P is a probability on $\mathcal{A}$ and P is $0 - 1$ valued. Note that we are still assuming that $\mathcal{A}$ is countably generated. Show that there is an atom A such that $P(A) = 1$. In particular if $\mathcal{A}$ is countably generated and has singleton sets, then any $0 - 1$ valued probability P must be concentrated at a point – that is, there is a point $a \in \Omega$ such that $P(A) = 1$ or 0 according as $a \in A$ or not.

15. If A is a Borel set in the real line and x is any real number, show that translate of A, defined by $A + x = \{y + x : y \in A\}$ is also a Borel set. Show that $-A = \{-y : y \in A\}$ is also a Borel set.

    Show that λ is translation invariant, this means for any Borel set A and any real number x, $\lambda(A) = \lambda(A + x)$.

    Show that if μ is a translation invariant measure on the Borel sets of real line with $\mu[0, 1] = 1$, then it must be the λ above.

    If I do not assume $\mu[0, 1] = 1$ then there are two possibilities : either $\mu[0, 1] < \infty$ OR $\mu[0, 1] = \infty$. In the first case show that μ is indeed a constant multiple of our good old λ. In the second case, show that $\mu(I) = \infty$ for every non-empty open interval.

In the second case there are several sub-possibilities. Here are some examples that satisfy translation invariance. Put $\mu(A) = 0$ or ∞ according as the set A is empty or not. Or put $\mu(A) = 0$ or ∞ according as the set A is countable or not. Or put $\mu(A)$ to be the number of points in the set A. We leave the story here. These are uninteresting as you will see after you learn integration.

16. Let μ be a finite measure on$(R, \mathcal{B})$. Put $F_\mu(x) = \mu(-\infty, x]$ for $x \in R$. Show that F_μ is (1) right continuous; (2) non-decreasing; (3) $\lim\limits_{x \to -\infty} F(x) = 0$ and $\lim\limits_{x \to \infty} F(x) < \infty$.

    Conversely given any such function F satisfying the three conditions above there is a finite measure μ on $(R, \mathcal{B})$ such that the given F is F_μ. Further this is a one-one correspondence between finite measures on $(R, \mathcal{B})$ and functions on R satisfying the three conditions above.

    If μ were Lebesgue measure and F is defined as above; what function would you get?

    Suppose μ is a measure on $(R, \mathcal{B})$ which is finite for bounded intervals. Such measures are called Radon measures. Define $F_\mu(x) = \mu(0, x]$ if $x \geq 0$ and $F_\mu(x) = -\mu(x, 0]$ if $x \leq 0$. Show that F_μ is (1) non-decreasing real valued (2) right continuous (3) $F(0) = 0$ function. Conversely, given any function F satisfying the three conditions above there is a Radon measure μ on $(R, \mathcal{B})$ such that the given F is F_μ. Further the correspondence between Radon measures on $(R, \mathcal{B})$ and functions satisfying the three conditions above is one-to-one.

    Every Radon measure is σ-finite. Do you think every σ-finite measure on $(R, \mathcal{B})$ is a Radon measure?

17. Here are some examples of sets and σ-fields.

    (1) Take Ω to be the set of all $n \times n$ matrices with complex entries. For $1 \leq i, j \leq n$, and intervals $U, V \subset R$; let $\Omega(i, j, U, V)$ be the set of all matrices whose (i, j)-th element has real part in U and imaginary part in V. The collection of all these sets generate a σ-field on Ω; called its Borel σ-field, denoted by $\mathcal{B}$.

    For a matrix $A = (a_{ij})$ we define its adjoint matrix, denoted by A^* as the matrix (b_{ij}) where $b_{ij} = \overline{a_{ji}}$. Here $\overline{x + iy} = x - iy$. Let H be the set of all Hermitian (also called self adjoint) matrices in Ω, that is, all matrices A such that $A = A^*$. Show that $H \in \mathcal{B}$.

    Let U_n be the set of all unitary matrices in Ω, that is, matrices A such that $AA^* = A^*A = I$. Here I is the $n \times n$ identity matrix, which has ones for its diagonal elements and zero for non-diagonal elements. Show $U_n \in \mathcal{B}$.

Let T be the set of upper triangular matrices in ω, that is matrices with all entries below the diagonal zero. Show $T \in \mathcal{B}$.

Show that the set of all invertible matrices is in $\mathcal{B}$.

(2) Let Ω be the set $C[0,1]$, the set of all real valued continuous functions on the interval $[0,1]$. For $0 \leq t \leq 1$ and an interval $I \subset R$, let $\Omega(t, I)$ be the set of all functions f in Ω such that $f(t) \in I$. The collection of all these sets generate a σ-field on Ω; called its Borel σ-field, denoted by $\mathcal{B}$.

Let $||f|| = \max\limits_{0 \leq t \leq 1} |f(t)|$. For any number $a \geq 0$, show that the set $\{f \in \Omega : ||f|| < a\}$ is in $\mathcal{B}$.

Show that the set $\{f \in \Omega : -8 \leq \int\limits_0^1 f(t)dt \leq 15\}$ is in $\mathcal{B}$.

(3) You must be wondering about the appearance of Borel σ-field at diverse places. Here is the common link. Let (X, d) be a metric space. Consider the σ-field generated by all the open subsets of X. This is called the Borel σ-field of X. Do you think the above two examples are special cases of this concept. What were the metric spaces?

18. Let us return to the real line. Every Borel set is nearly an open set. What does this mean?

Suppose that P is a probability on $(R, \mathcal{B})$. Show that given a Borel set B and a number $\epsilon > 0$, we can get a closed set C and an open set U such that $C \subset B \subset U$ and $P(U - C) < \epsilon$. (Show that the class of sets for which you can do this is a σ-field.) In particular, for any Borel set A, $P(A) = \sup\{P(K) : K \text{ compact }, K \subset A\}$ and $P(A) = \inf\{P(U) : U \text{ open}, U \supset A\}$.

As a consequence, given any Borel set $B \subset R$ and $\epsilon > 0$, you can get open set $U \supset B$ such that $P(U - B) < \epsilon$, that is, you can get an open set as close (in probability) to B as desired.

Show that the above statement is true for any finite measure.

Assume that μ is a Radon measure on $(R, \mathcal{B})$. Show that for any Borel set A, $\mu(A) = \sup\{\mu(K) : K \text{ compact }, K \subset A\}$ and $\mu(A) = \inf\{\mu(U) : U \text{ open}, U \supset A\}$.

Do you think (do not spend too much time) the statement holds for any σ-finite measure on $(R, \mathcal{B})$?

Remember, if you are stuck with a problem, you ask yourself: can I do the simplest case?

Here are some trivial, but useful, facts. You should be able to complete all but the last exercise in few minutes. While reading measure theory, you should keep in mind that our intent was modest: to define length for more general sets than intervals. In the process, we got much more than what we were looking for — a general theory. The theory itself is not difficult or abstract, as is usually made out. It is fundamental and useful. The course would cover enough to justify these adjectives.

19. Let F be a (Radon) distribution function on R and μ the corresponding measure.

    Can we explain $\mu(\{5\})$ by looking at F? Yes. For any number $a \in R$; $\mu(\{a\}) = F(a) - F(a-)$. Here $F(a-) = \lim_{n\to\infty} F(a - \frac{1}{n})$. Prove this. If $\mu(\{a\}) = \alpha$ it is customary to say that μ puts mass α at the point a. Of course if $\alpha = 0$, one says that μ does not put any mass at a.

    In particular, $\mu(\{a\}) = 0$ iff F is continuous at a. Measures on R which give mass zero to all singleton sets are called continuous measures. Thus a Radon measure is continuous iff its distribution function is continuous.

    The correspondence between Radon measures and Radon diistribution functions tells us $\mu(a, b] = F(b) - F(a)$. Can we explain $\mu(a, b)$? Yes, show $\mu(a, b) = F(b-) - F(a)$. Show $\mu[a, b) = F(b-) - F(a-)$.

    Let us enumerate the set of rationals as a sequence $r_1, r_2, \cdots$. Consider the Radon measure, actually a probability measure μ defined for $B \in \mathcal{B}$ by
    $$\mu(B) = \sum_{r_n \in B} \frac{1}{2^n}.$$

    In other words it corresponds to the measure that puts mass $1/2^n$ at the rational r_n. Its distribution function is a real valued function on R which is continuous at all irrationals but discontinuous at all rationals.

    It is an interesting non-trivial exercise in analysis to show that you can not find a real valued continuous function on R which is continuous at all rationals but discontinuous at all irrationals.

20. It is a general mathematical nicety that if a set has a mathematical structure, then certain subsets which are closed under the necessary operations can be regarded as new structures — substructures of the original one. Here are two examples. If G is a group and H is a subset of G closed under the group operation (binary) and inverse (unary), then you have a new group H, a subgroup of G. If V is a vector space and W is a subset of

V closed under addition and scalar (whatever be the field) multiplication, then you have a new vector space W, a subspace of V.

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Let $X \in \mathcal{B}$ with $\mu(X) > 0$. Show that $\mathcal{C} = \{C \subset X : C \in \mathcal{B}\}$ is a σ-field of subsets of X. Define ν by $\nu(C) = \mu(C)$ for sets $C \in \mathcal{C}$. Show that $(X, \mathcal{C}, \nu)$ is a measure space. This is called restriction of μ to the set $X \in \mathcal{B}$.

Show that X is a finite measure space if Ω is so, though not conversely. Show that X is σ-finite measure space if Ω is so, though not conversely.

Here are concrete situations. You can consider $(R, \mathcal{B}, \lambda)$, Lebesgue measure on the real line, but you may be interested only in numbers between zero and one. The you take $X = [0, 1]$. There is no need to hang on to all the real numbers, you can as well forget everything beyond $[0, 1]$, that is look at, the above construct with $X = [0, 1]$.

If your world consists of probability spaces, you start with a probability space $(\Omega, \mathcal{B}, \mu)$. Of course X, as constructed above, need not be a probability space. One usually normalizes, put $\mu_X(C) = \mu(C)/\mu(X)$ for $C \in \mathcal{C}$. This makes sense because we assumed $\mu(X) > 0$. Show that $(X, \mathcal{C}, \mu_X)$ is again a probabilty space.

21. As above let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Earlier, we restricted to a subset X of Ω; where $X \in \mathcal{B}$ and in a sense looked at all of $\mathcal{B}$ restricted to X. We can also keep all of Ω but restrict $\mathcal{B}$. Here is how.

    Let $\mathcal{C} \subset \mathcal{B}$ such that $\mathcal{C}$ is itself a σ-field on Ω. Put $\nu(C) = \mu(C)$ for $C \in \mathcal{C}$. Then $(\Omega, \mathcal{C}, \nu)$ is a measure space. Prove this. You may say ν is just μ; you are nearly correct except that the doman of definition of ν is only sets that belong to $\mathcal{C}$.

    To see a concrete example, again consider $(R, \mathcal{B}, \lambda)$. Let $\mathcal{C}$ be the collection of symmetric Borel sets. More precisely, say that a Borel set B is symmetric if $x \in B \Rightarrow -x \in B$. Let $\mathcal{C}$ be the collection of symmetric Borel sets. Show that this class is a σ-field on R. Or to look at another example, let Z be the set of integers. Say that a Borel set B is Z-invariant if $x \in B \Rightarrow \forall n \in Z, x + n \in B$. Let $\mathcal{C}$ consist of all Z-invariant Borel sets.

    Do you see some interplay between measures and group actions.

22. It is again a general mathematical nicety that maps between sets facilitate export/import of mathematical structures between them. Here are two examples. If G_1 is a group and T is a one-to-one map on G_1 onto a set G_2, then you can bring forward the group structure from G_1 to G_2. If T is a linear map from a vector space V_1 to another vector space V_2 (both, say, over real field), any linear functional on V_2 can be taken backward to V_1. Basically, these constructs are sponsored by simpe (but appropriate) composition of maps.

Let Ω_1 be a set with a σ-field, say, $\mathcal{B}_1$. Suppose $T : \Omega_1 \to \Omega_2$ is a map, not necessarily one-one, not necessarily onto. We can define a σ-field on Ω_2 as follows.

$$\mathcal{B}_2 = \{B \subset \Omega_2 : T^{-1}(B) \in \mathcal{B}_1\}.$$

Recall that for a subset $B \subset \Omega_2$; $T^{-1}(B)$ is defined as the set of points $\omega_1 \in \Omega_1$ such that $T(\omega_1) \in B$. Show that the collection $\mathcal{B}_2$ is indeed a σ-field of subsets of Ω_2.

Let now μ_1 be a measure on $\mathcal{B}_1$. Put $\mu_2(B) = \mu_1(T^{-1}(B))$ for $B \in \mathcal{B}_2$. This is meaningful because for $B \in \mathcal{B}_2$, we know $T^{-1}(B) \in \mathcal{B}_1$ and hence its μ_1 value can be calculated. Show that μ_2 is indeed a measure on $\mathcal{B}_2$.

If T is not onto, then any subset of Ω_2 which is disjoint from range of T is in $\mathcal{B}_2$ and gets μ_2 value zero. If μ_1 is finite so is μ_2. Is this statement true if 'finite' is replaced by 'σ-finite'?

23. In the above construction sometimes the set Ω_2 comes equipped already with its own σ-field.

    Let $(\Omega_1, \mathcal{B}_1, \mu_1)$ be a measure space. Let $\mathcal{B}_2$ be a σ-field of subsets of Ω_2. Let $T : \Omega_1 \to \Omega_2$ such that for every $C \in \mathcal{B}_2$ we have $T^{-1}(C) \in \mathcal{B}_1$. Then we can rightaway define $\mu_2(C) = \mu_1(T^{-1}(C))$ for $C \in \mathcal{B}_2$. Show that μ_2 is a measure on $(\Omega_2, \mathcal{B}_2)$.

    Here is a concrete example. Take $\Omega_1 = [0, 2]$; $\mathcal{B}_1$ to be the collection of Borel sets in R which are contained in $[0, 2]$, μ_1 is Lebesgue measure. Take $\Omega_2 = [0, 4]$; $\mathcal{B}_2$ to be the collection of Borel sets in R which are contained in $[0, 4]$. Take $T(x) = x^2$. The map T is too nice, you can immediately see that inverse image of an interval is again an interval. Deduce from this that T fulfils the condition mentioned above. Thus we get μ_2 on $[0, 4]$. Does it have anything to do with the Lebesgue measure on $[0, 4]$. Yes, in fact it is this relation that leads to your 'change of variable' formula, or method of substitution in calculus. We shall return to this point later.

24. After those general constructs, let us return to real line. A subset C of reals is called a Cantor set, if it is non-empty, compact, does not contain any interval, every point of C is a limit point of C. For example usual cantor set in $[0, 1]$ is *a* Cantor set. Of course, by usual homeomorphisms between intervals, you can say that every non-degenerate interval contains a Cantor set.

    Let C be a Cantor set. Let a and b be the inf and sup of C so that $a \in C$, $b \in C$ and $C \subset [a, b]$. Show that $[a, b] - C$ (complement of C in $[a, b]$) is a countable disjoint union of open intervals. Denote this family of open intervals by $\mathcal{I}$. For I_1 and I_2 in $\mathcal{I}$ say $I_1 < I_2$ if $\alpha < \beta$ where $\alpha \in I_1$ and $\beta \in I_2$. Show that this is a good definition.

Show that the set $\mathcal{I}$ is linearly ordered. Show that $\mathcal{I}$ has neither a first element nor a last element — in the order we defined. Show that if $I_1 < I_2$ then there is an $I_3 \in \mathcal{I}$ such that $I_1 < I_3 < I_2$. In other words there are no breaks in this ordered set. Such a linearly ordered set is named 'dense-in-itself without end points'.

Let A and B be two countable linearly ordered sets each of which is 'dense-in-itself without end points'. A celebrated theorem of Cantor says that we can find an isomorphism between them, that is, there is a map $\varphi : A \to B$ which is one-one, onto, order preserving. Here is the idea, loosely expressed. Start with any enumeration $A = \{a^1, a^2, \cdots\}$ and $B = \{b^1, b^2 \cdots\}$. We re-enumerate $A = \{a_1, a_2, \cdots\}$ and $B = \{b_1, b_2, \cdots\}$ so that the map sending a_i to b_i will serve our purpose. Put $a_1 = a^1$ and $b_1 = b^1$. Put $a_2 = a^2$. Depending on how it is related to a_1 pick up the first b^i which is related to b_1 the same way. Name it b_2. Now take the first b^i which you have not used, name it b_3, see how it is related to b_1, b_2 and pick up the first a^i which is related to a_1, a_2 in the same way, name it a_3. Look at the first unused a^i, name it a_4, get b_4. Get the idea first and then execute, alternating between a's and b's.

Let C_1 and C_2 be two Cantor sets. Show that they are homeomorphic. In fact there is a homeomorphism of R which brings C_1 to C_2.

Show that there is a probability on R which is supported on the usual Cantor set. Here 'supported' just means it gives value zero to the complement of the Cantor set. Idea: Look at Cantor distribution function of Ex set 1.

Given any Cantor set show that there is a continuous probability on R supported by that Cantor set.

Show that there is a sequence of disjoint Cantor sets $C_1, C_2, \cdots$ such that-every non-empty open interval contains one of these. Idea: start with intervals with rational end points and proceed.

Show that there is a continuous σ-finite measure on $(R, \mathcal{B})$ which gives infinite measure for every interval. If we did not want continuous measure with this property it is easy. Just put unit mass at each rational, in other words, take $\mu(B) =$ number of rationals in B.

In particular this is a continuous σ-finite measure on R which is NOT a Radon measure. This is also an example of a continuous σ-finite measure for which the statement 'every Borel set is nearly open' fails. More precisely, $\mu(B) = \inf\{\mu(U) : U \text{ open } B \subset U\}$ fails for some Borel sets.

25. Let Ω be a countable set and Π be a partition of Ω. If $\mathcal{A}$ is the collection of all subsets of Ω which are union of sets from the partition, show that $\mathcal{A}$ is a σ-field. Conversely, if $\mathcal{A}$ is a σ-field of subsets of Ω, show that there is indeed a partition of Ω such that the given σ-field consists of all subsets of Ω which can be expressed as unions of sets in the partition. Do you think this last statement is correct when Ω is not countable.

    In particular, show that if a σ-field is not finite, then it must have at least as many sets as there are points in the interval $[0, 1]$. In other words, the cardinality of an infinite σ-field is at least $2^{\aleph_0}$ (What the hell is this symbol?).

26. Let Ω be an uncountable set and $\mathcal{A}$ be the countable-cocountable σ-field. If f is a real measurable function, show that there is a countable $S \subset \Omega$ and a real number a such that $f(\omega) = a$ for all $\omega \notin S$. Conversely, if f is any such function, then show that it is a measurable function.

27. Consider $(R, \mathcal{B})$. show that every real valued continuous function is measurable. Show that every left continuous (or right continuous) function is measurable. Show that every monotone function is measurable.

    Let f be *any* real valued function on R. Show that

$$S = \{x : f \text{ is continuous at } x\} \in \mathcal{B}.$$

28. Sometimes a σ-field does not contain a set we are interested in. No problem, throw it in!

    Let $\mathcal{A}$ be a σ-field of subsets of Ω. Let $M \subset \Omega$ and $M \notin \mathcal{A}$. We want to describe the smallest σ-field which contains $\mathcal{A}$ and M, denoted by $\mathcal{A}'$. Suppose I take two sets A and B from $\mathcal{A}$. Then show that the set $(A \cap M) \cup (B \cap M^c) \in \mathcal{A}'$. Conversely, show that every set in $\mathcal{A}'$ is like this.

    More generally, let $\{M_n : n \geq 1\}$ be a partition of Ω; the partiiton could be finite or countably infinite. How do sets in the σ-field $\mathcal{A}'$, generated by $\mathcal{A}$ and this family $(M_n : n \geq 1)$ look like? First show that if we take sets $\{A_n : n \geq 1\}$ in $\mathcal{A}$, then the set $\cup_{n\geq 1}(A_n \cap M_n)$ is in $\mathcal{A}'$. Show that every set in $\mathcal{A}'$ is like this. Observe the special case $\{M, M^c\}$ corresponds to earlier paragraph.

29. Let us return to measures. Suppose $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space. There are some sets, which may not be in our σ-field but their measure is *obvious*. For example if $N \in \mathcal{A}$ and $\mu(N) = 0$, then for every subset $M \subset N$, clearly $\mu(M) = 0$. The reason this statement, as it stands, is

meaningless is that the set M may not be in our σ-field and hence we can not talk about its μ value. We can rectify the situation as follows.

Let

$$\mathcal{I} = \{M : \exists N \in \mathcal{A}, \mu(N) = 0; M \subset N\}.$$

Thus $\mathcal{I}$ consists of subsets of sets of measure zero. Show that $\mathcal{I}$ is a σ-ideal, which means the following: it is closed under subsets and countable unions. Let

$$\mathcal{A}_0 = \{A \Delta M : A \in \mathcal{A}; M \in \mathcal{I}\}.$$

Show that $\mathcal{A}_0 \supset \mathcal{A}$ and is a σ-field. If $B \in \mathcal{A}_0$ and $B = A\Delta M$ with $A \in \mathcal{A}$ and $M \in \mathcal{I}$, then put $\tilde{\mu}(B) = \mu(A)$. Show that this is a good definition.

Show that $\tilde{\mu}$ is a σ-finite measure on $\mathcal{A}_0$ and it extends μ. This σ-finite measure space $(\Omega, \mathcal{A}_0, \tilde{\mu})$ is called the *completion* of the measure space $(\Omega, \mathcal{A}, \mu)$. Given $A \in \mathcal{A}_0$ show that there are sets A_i and A_e in $\mathcal{A}$ such that $A_i \subset A \subset A_e$ and $\mu(A_i) = \mu(A_e)$.

Suppose that someone gives us a measure space $(\Omega, \mathcal{B}, \nu)$ which extends $(\Omega, \mathcal{A}, \mu)$. Here 'extends' simply means, $\mathcal{B} \supset \mathcal{A}$ and for sets $A \in \mathcal{A}$ we have $\nu(A) = \mu(A)$. Then show that $\tilde{\mu}$ and ν agree on $\mathcal{B} \cap \mathcal{A}_0$

Sometimes we denote $\tilde{\mu}$ by just μ. This is justified, because everyone agrees for measure of sets in $\mathcal{A}_0$ — of course, technically the domain of definition of μ is $\mathcal{A}$ where as the domain of definition of $\tilde{\mu}$ is $\mathcal{A}_0$, this causes no confusion.

30. A measure space $(\Omega, \mathcal{C}, \nu)$ is said to be 'complete' if

$$N \in \mathcal{C}, \nu(N) = 0, M \subset N \Rightarrow M \in \mathcal{C}.$$

Of course in such a case automatically, $\nu(M) = 0$. Sometimes one also says $\mathcal{C}$ is complete under ν.

Show that the completion $(\Omega, \mathcal{A}_0, \mu)$ as defined in the earlier exercise is indeed a complete measure space.

31. Let us now return to the Caratheodory extension theorem and settle a tantalizing question. To fix ideas (though there is no need to!) let us consider $\Omega = R$, $\mathcal{F}$ be the field of finite disjoint union of intervals of the form $(a, b]$; and μ be a probability measure on $\mathcal{F}$. We extended it to $\mathcal{B} = \sigma\{\mathcal{F}\}$ as a measure.

There were three natural steps. First a modest extension to the class of sets $\mathcal{F}_\sigma$. Second, to get an idea of the maximum value we can give to set B, namely $\mu^*(B) = \inf\{\mu(A) : A \in \mathcal{F}_\sigma\}$. Third step is to collect the class $\mathcal{A}$ of all those sets B for which we have no decision to make, that is, $\mu^*(B) + \mu^*(B^c) = 1$. Then $\mathcal{A}$ is a σ-field; μ^* is a probability on it (denoted by μ) and it extends μ. Finally we said, ok, restrict μ^* to just $\mathcal{B}$ because we know $\mathcal{B} \subset \mathcal{A}$.

The question is: having got extension to a large class $\mathcal{A}$ of sets, is it not foolish to just restrict to $\mathcal{B}$. One answer is that this class $\mathcal{A}$ depended on our μ and there is no neat way to prejudge and state the class to which we are extending the measure. On the other hand $\mathcal{B} = \sigma\{\mathcal{F}\}$ does not depend on μ and acts as a reference family for extending *any* probability on $\mathcal{F}$.

The question remains; Did we loose any information by such a restriction? No. This is because your $(R, \mathcal{A}, \mu)$ is nothing but the completion of $(R, \mathcal{B}, \mu)$. Here is the argument. I am using μ in place of $\tilde{\mu}$ because you should get used to such a thinking.

Show that for any $B \subset \Omega$, there is $B_e \supset B$; $B_e \in \mathcal{F}_\sigma$ such that $\mu^*(B) = \mu(B_e)$. Deduce that for any $B \in \mathcal{A}$, there are sets $B_i \subset B \subset B_e$ such that $B_i, B_e \in \mathcal{B}$ with $\mu(B_i) = \mu(B) = \mu(B_e)$. Deduce that B is in the completion of $(R, \mathcal{B}, \mu)$.

Show that if $\mu^*(B) = 0$, then $B \in \mathcal{A}$. Deduce that every set in the μ-completion of $\mathcal{B}$ is in $\mathcal{A}$.

32. Let us return to the first part of Exercise 28. Suppose we had a probability on $(\Omega, \mathcal{A})$. Can we extend it to a probability on $\mathcal{A}'$? Yes. Here is how.

    Let $M \notin \mathcal{A}$. Put

$$\overline{\mu}(M) = \inf\{\mu(B) : B \supset M; B \in \mathcal{A}\};$$

$$\underline{\mu}(M) = \sup\{\mu(B) : B \subset M; B \in \mathcal{A}.$$

    These sup and inf are actually max and min. In other words, show that there are sets $M_i, M_e \in \mathcal{A}$; $M_i \subset M \subset M_e$ with $\underline{\mu}(M) = \mu(M_i)$ and $\overline{\mu}(M) = \mu(M_e)$.

    If $\underline{\mu}(M) == \overline{\mu}(M)$, then show that M is in the completion of $\mathcal{A}$ and this leads to promised extension, since $\mathcal{A}'$ is then contained in the completed σ-field.

    Let $\underline{\mu}(M) < \overline{\mu}(M)$. We shall now show that all values between $\underline{\mu}(M)$ and $\overline{\mu}(M)$ are possible values for the measure of M.

    First, suppose that ν on $\mathcal{A}'$ extends μ. Then we must necessarily have $\underline{\mu}(M) \leq \nu(M) \leq \overline{\mu}(M)$.

    Towards the converse, take any number c such that $\underline{\mu}(M) \leq c \leq \overline{\mu}(M)$. We show an extension ν with $\nu(M) = c$. Denote $\underline{\mu}(M) = \alpha$ and $\overline{\mu}(M) = \beta$, so that $\alpha \leq c \leq \beta$. If $B \in \mathcal{A}'$, show that we can express

$$B = [B_1 \cap (M - M_i)] \;\cup\; [B_2 \cap (M_e - M)] \cup B_3;$$

where

$$B_1, B_2, B_3 \in \mathcal{A}; B_1, B_2 \subset M_e - M_i; B_3 \subset M_i \cup M_e^c.$$

Moreover, the set B_3 is uniquely determined by B. The sets B_1 and B_2 may not be uniquely determined, but the numbers $\mu(B_1)$ and $\mu(B_2)$ are uniquely determined. In otherwords, you can express B as the above union with sets B_1, B_2, B_3 as well as with sets B_1', B_2', B_3' satisfying the above conditions, then $B_3 = B_3'$; $\mu(B_1) = \mu(B_1'); \mu(B_2) = \mu(B_2')$. Put

$$\nu(B) = \frac{c-\alpha}{\beta-\alpha}\mu(B_1) + \frac{\beta-c}{\beta-\alpha}\mu(B_2) + \mu(B_3).$$

Show that ν is a probability on $\mathcal{A}'$; ν extends μ; and $\nu(M) = c$.

We could have done this argument with finte measures instead of probability. We could do with σ-finite measures too with a careful formulation. But let us not make things boring.

33. Let us return to Exercise 29.

Can we say anything beyond $\mathcal{A}_0$. In other words, can we agree on measure for sets belonging to a σ-field larger than $\mathcal{A}_0$? No.

Before doing let me draw your atttention to a wrong impression you may entertain. Afterall, if $A \in \mathcal{A}$ and $\mu(A) = \infty$, then do we not all agree that any superset of A must have measure ∞. Should we then — as we did for sets of measure zero — enlarge $\mathcal{A}$ by including all such supersets of sets of infinite measure and declare their measure to be ∞. NO. (why?)

Use the notation of exercise 29, except that we assume now μ is a probability. Let $M \notin \mathcal{A}_0$. Put

$$\overline{\mu}(M) = \inf\{\mu(B) : B \supset M; B \in \mathcal{A}_0\} = \inf\{\mu(B) : B \supset M; B \in \mathcal{A}\};$$

$$\underline{\mu}(M) = \sup\{\mu(B) : B \subset M; B \in \mathcal{A}_0\} = \sup\{\mu(B) : B \subset M; B \in \mathcal{A}\}.$$

These sup and inf are actually maximum and minimum.

If $\underline{\mu}(M) == \overline{\mu}(M)$, then show that $M \in \mathcal{A}_0$.

Since we have assumed that $M \notin \mathcal{A}_0$, we must have $\underline{\mu}(M) < \overline{\mu}(M)$.Show that all values between $\underline{\mu}(M)$ and $\overline{\mu}(M)$ are possible values for the measure of M. In other woords, if $(\Omega, \mathcal{B}, \nu)$ is a measure space extending $(\Omega, \mathcal{A}_0, \mu)$ such that $M \in l\mathcal{B}$. Then we must necessarily have $\underline{\mu}(M) \leq \nu(M) \leq \overline{\mu}(M)$. Conversely, if we take a number c such that $\underline{\mu}(M) \leq c \leq \overline{\mu}(M)$ then there is an extension ν with $\nu(M) = c$ to the σ-field $\mathcal{B}$ generated by $\mathcal{A}_0$ and M.

34. Show that the function

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-1/x^2} & \text{for } x \geq 0 \end{cases}$$

is a C^∞-function, that is, it has derivatives of all orders.

Let $a \in R$. Show that there is a C^∞-function which is zero on $(-\infty, a)$ and strictly positive on (a, ∞).

Let $b \in R$. Show that there is a C^∞-function which is strictly positive on $(-\infty, b)$ and zero on (b, ∞).

Let $a < b \in R$. Show that there is a C^∞-function which is strictly positive on (a, b) and zero outside.

Let $a < b \in R$. Show that there is a C^∞-function f such that $f(x) = 0$ for $x < a$; $f(x) = 1$ for $x > b$ and in between increases from zero to one. (Use indefinite integral).

Let $a < b \in R$ and $\epsilon > 0$. Show that there is a C^∞-funciton f such that $f(x) = 0$ for $x < a - \epsilon$; increases from zero to one during $a - \epsilon$ to a; $f(x) = 1$ for $a \leq x \leq b$; then decreases to zero during b to $b + \epsilon$; $f(x) = 0$ for $x > b + \epsilon$.

Given any bounded interval $[a, b]$, show that there is a sequence of C^∞-functions which converge bp to its indicator.

35. Consider $L^1(R)$. If no reference measure is specified, as we did now, it is understood that we are considering the Lebesgue measure.The distance is $d(f, g) = \int |f - g| d\lambda$. Show that measurable functions with compact support are dense.

Show that bounded measurable functions with compact support are dense.

Show that simple functions with compact support are dense.

show that simple functions based on intervals, that is, finite linear combinations of indicators of intervals, are dense.

Show that C^∞-functions with compact support are dense. In particular, continuous functions with compact support are dense, or, C^2 functions with compact support are dense in L^1.

Is it necessary to assume in the above discussion that we had Lebesgue measure. Show that it is true for any Radon measure.

For a general σ-finite measure, NO conitnuous function — except the zero function — may be in L^1!

36. Consider $L^1(\Omega, \mathcal{A}, \mu)$. Show that simple functions based on sets in $\mathcal{A}$, that is finite linear combinations of indicators of sets in $\mathcal{A}$, are dense.

Let us now specialize, assume that μ is a finite measure. Let $\mathcal{F}$ be a field generating $\mathcal{A}$. Show that simple functions based on sets in $\mathcal{F}$ are dense.

Do not assume that μ is finite, but it is σ-finite on $\mathcal{F}$. Show that the above statement is still correct, in the sense, simple functions based on sets of finite measure in $\mathcal{F}$ are dense. If you take a set of infinite measure, its indicator is not in the L^1 space, hence this new adjective. Would this statement be true if μ is σ-finite on $\mathcal{A}$ instead of on $\mathcal{F}$?

It is useful to know dense subsets of topological spaces. Remember how our life depended on the set of rationals in understanding real numbers and in the analysis of functions on R.

37. In topology, first category sets are supposed to be small, whereas in measure theory sets of measure zero are supposed to be small. Recall, a closed set $A \subset R$ is said to be nowhere dense if it contains no interval of positive length. A set $A \subset R$ is said to be of first category if we can express $A = \cup A_n$ where each $\overline{A}_n$ is nowhere dense. Here the overline denotes closure.

Fix $\epsilon > 0$; enumerate set of rationals; take open interval of length $\epsilon/2^n$ centered around the n-th rational and let A_ϵ be the union of these open intervals. Let $A = \cap A_{1/k}$.

Show that A has Lebesgue measure zero (thus measure theoretically, A^c is a full set) where as A^c is of first category (thus, A^c is small topologically).

38. Suppose that $A \subset R$ be bounded with $\lambda(A) > 0$. Show that there is an open interval I such that $\lambda(A \cap I) \geq \frac{3}{4}\lambda(I)$, that is at least three-fourths of the interval is our set A.

If $A \subset R$ is Borel with positive Lebesgue measure, then show that the difference set $\{x - y : x, y \in A\}$ contains a non-degenerate interval around zero.

A set $A \subset R$ is said to be Lebesgue measurable if it is in the completion of the Borel σ-field w.r.t. Lebesgue measure. Equivalently, it is in the class $\mathcal{A}$ that Caratheodory gave us while extending length measure from the basic field of finite disjoint unions of leftopen-rightclosed intervals. Equivalently, Lebesgue measurable set is a set for which there is no dispute about its length. The class of Lebesgue measurable sets is denoted by $\mathcal{L}$. Show that the statement of earlier para is true for $A \in \mathcal{L}$.

39. For two numbers $x, y \in R$, say $x \sim y$ if $x - y$ is rational. Show that this is an equivalence relation. Each equivalence class is countable. Let us pick one point from each equivalence class and name the resulting set M.

Can the difference set of M contain a rational other than zero? If M were Lebesgue measurable can M have positive measure.

You have earlier shown that λ is translational invariant on $\mathcal{B}$, do the same for $\mathcal{L}$.

Show that R is a countable union of translates of M. If M were Lebesgue measurable, can it have zero measure?

Can M be Lebesgue measurable?

Can we make a set by picking points from each of the uncountably many equivalence classes? Obvoiusly, it depends on our rules for making sets. We assume we can do it. This principle of making a set by picking one point from each of a collection of disjoint sets is called 'Axiom of Choice'. We are free to disbelieve it, but then Robert Solovay showed that it is quite possible that every set is Lebesgue measurable.

40. It is time to understand integration. Consider $(R, \mathcal{B})$.

    Fix $a \in R$. Consider point mass at a, that is, consider the measure μ, defined as $\mu(B) = 1$ or 0 according as $a \in B$ or not. Show that this is a measure. Show that every real measurable function is integrable and $\int f d\mu = f(a)$. This measure is also denoted by δ_a.

    Consider k distinct real numbers $a_i; 1 \leq i \leq k$ and positive numbers $w_i; 1 \leq i \leq k$. Consider the measure which puts mass w_i at a_i. That is $\mu(B) = \sum_{i:a_i \in B} w_i = \sum_{i=1}^{k} w_i I_B(a_i)$. Show that this is a measure. Show that every real measurable function is integrable and $\int f d\mu = \sum_{i=1}^{k} w_i f(a_i)$. This measure is also denoted as $\sum w_i \delta_{a_i}$.

    Consider distinct real numbers $a_i : i \geq 1$ and nonnegative numbers $w_i : i \geq 1$. Put mass w_i at a_i. That is the measure $\mu(B) = \sum_{i:a_i \in B} w_i = \sum_{i \geq 1} w_i I_B(a_i)$. Show that this is a measure. Show that f is integrable iff $\sum w_i |f(a_i)| < \infty$ and in that case $\int f d\mu = \sum w_i f(a_i)$.

    Go through the details, we discussed, connecting Riemann integral (learnt in Calculus) and Lebesgue integral (learnt now).

41. Just to see a different example, put $F(x) = 0$ for $x < 0$; $F(x) = 1 + x$ for $0 \leq x \leq 1$; $f(x) = 2$ for $x \geq 1$. This is right continuous non-decreasing function. So this corresponds to a unique measure μ on $(R, \mathcal{B})$.

    Show that, $\mu(\{0\}) = 1$ and $\mu(\{x\}) = 0$ for all $x \neq 0$. Show that $\mu(-\infty, 0) = 0$ and $\mu(1, \infty) = 0$. Show that for any interval $(a, b) \subset (0, 1)$, $\mu(a, b) = b - a$.

    Thus μ is sum of two things: point mass at zero and Lebesgue measure on $(0, 1)$. Show that integral is also sum of corresponding integrals, that is, for bounded measurable f, $\int f d\mu = f(0) + \int f d\lambda_1$, where λ_1 is the Lebesgue measure restricted to $(0, 1)$.

42. If μ and ν are two measures on the same space $(\Omega, \mathcal{A})$, then we know that $\mu + \nu$ is also a measure. Show that a measurable function is $(\mu + \nu)$-integrable iff it is μ-integrable and ν-integrable. In such a case $\int f d(\mu + \nu) = \int f d\mu + \int f d\nu$.

    Let $c > 0$ and $\nu = c\mu$. Show that f is ν-integrable iff it is μ-integrable and in that casse $\int f d\nu = c \int f d\mu$.

I have followed Jacques Neveu 'Foundations of the Calculus of Probability' for constructing measures and Rudin 'Real and Complex Analysis' for integration. As I said, you can consult any book on measure theory.

Now that we have built a firm theory of integration, it is time to recall what the architect Lebesgue says in the introduction to his book 'Lecons sur L' intégration'.

One might ask if there is sufficient interest to occupy oneself with such complications, and if it is not better to restrict oneself to the study of functions that necessitate only simple definitions. .... As we shall see, in this course, we would then have to renounce the possibility of resolving many problems posed long ago, and which have simple statements. It is to solve these problems, and not for the love of complications, that I have introduced in this book a definition of the integral more general than that of Riemann.

Without going into the finer details, here are our beginnings. Riemann integral was the main character till about 1900. Motivated by his studies on the patterns of digits in decimal expansion, Emile Borel, in 1898, extended the concept of length for more general sets than intervals and their finite unions. Henri Lebesgue, around 1902, extended length to all Borel sets and defined the Lebesgue integral, thus laying a firm foundation for an epoch making general theory of integration. Soon after in 1905, Vitali constructed his non-measurable set showing certain limitations. Radon realized that the theory can be pushed to more general situations than length, but still living on the Real line. Maurice Frechet around 1915 realized that what you need is a nice class of sets and a nice set function on that class of sets to carry out integration. In 1918, Caratheodory developed his theory of extending measure from a field to σ-field, thus laying a firm foundation for constructing measures. The theorems of F. Riesz (representation of linear functionals), Haar (translation invariant measures on locally compact groups) and developments in functional Analysis/Harmonic Analysis made the theory an essential part of life. S. Saks, Von Neumann and others (gave finishing touches?) brought the theory to the masses (I mean, popularized through lectures). Around 1923 A N Kolmogorov made measure theory as the foundation for a rigorous development of Probability Theory.

43. There are some trivial things that you need to take note of.

(a) Let Ω be a set. Consider the σ-field $\mathcal{A} = \{\emptyset, \Omega\}$. Show that measurable functions are just the constant functions.

Let us take a subset $B \subset \Omega$ and put $\mathcal{A} = \{\emptyset, B, B^c, \Omega\}$. Show that a function is measurable iff it is constant on B and constant on B^c. (need not be the same constant on both.)

Generalize when $\mathcal{A}$ is generated by finitely many sets.

(b) Let $(\Omega, \mathcal{A})$ be a *measurable space*, that is a non-empty set and a σ-field on it. Suppose that $x, y \in \Omega$. Suppose that for every real measurable function f, we have $f(x) = f(y)$. Then show that for every set $A \in \mathcal{A}$; $x \in A \leftrightarrow y \in A$.

Conversely suppose that for every set $A \in \mathcal{A}$; $x \in A \leftrightarrow y \in A$. Then show that for every real measurable f; we must have $f(x) = f(y)$.

(c) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{B} \subset \mathcal{A}$ be a subσ-field. That is, $\mathcal{B}$ is a σ-field and every set in $\mathcal{B}$ is already in $\mathcal{A}$. Let ν be the restriction of μ to $\mathcal{B}$, that is for $B \in \mathcal{B}$; we put $\nu(B) = \mu(B)$. We know that ν is a measure on $\mathcal{B}$. Observe that every $\mathcal{B}$ measurable real valued function f is also $\mathcal{A}$ measurable. Show that f is ν-integrable iff it is μ-integrable and then $\int f d\nu = \int f d\mu$.

(d) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Fix a set $S \in \mathcal{A}$. Let us consider the new measure ν defined by $\nu(A) = \mu(A \cap S)$. Thus ν is same as μ restricted to the set S. Show that a measurable function f is ν integrable iff fI_S is μ integrable and then $\int f d\nu = \int f I_S d\mu$.

44. In Ex 18, we saw one of the three basic principles of Littlewood, namely, every Borel subset of R is nearly an open set. We shall now discuss his second principle: every Borel function on R is nearly a continuous function. This means the following: Given a Borel function f and $\epsilon > 0$, there is a continuous function φ such that $\lambda\{x : f(x) \neq \varphi(x)\} \leq \epsilon$. Denote this statement by (*).

Given a closed set $F \subset R$, show that $g_F(x) = \min\{|x - a| : a \in F\}$ is a continuous function on R.

Given a closed set F and an open set U with $F \subset U$, there is a continuous function φ such that $\varphi(x) = 1$ for $x \in F$; $\varphi(x) = 0$ for $x \notin U$ and $0 \leq \varphi \leq 1$. Put $H = U^c$ and try $\varphi = g_H/(g_F + g_H)$.

Show that (*) holds for the indicator of a bounded Borel set with the corresponding φ also taking values between zero and one.

Show that (*) holds for any simple function f with bounded support with the corresponding φ also having values between $\min f$ and $\max f$.

Show that (*) holds for any bounded function f with bounded support. If f is zero outside $[a, b]$, express $f = \sum f_n$ with each f_n simple, supported on $[a, b]$, and $\sum_n ||f_n|| < \infty$ where $||f_n|| = \sup_x |f_n(x)|$. Get φ_n as above with $\epsilon/2^n$ for f_n; then $\varphi = \sum \varphi_n$ is meaningful and does the job.

Show that (*) holds for any Borel function with bounded support.Set $f_k(x) = f(x)$ if $|f(x)| \leq k$; $f_k(x) = k$ if $f(x) \geq k$; $f_k(x) = -k$ if $f(x) \leq -k$. Show that taking k large and carefully approximating f_k would do for f as well.

Show that (*) holds for any Borel f. Consider $f_n = f I_{[n,n+1]}$ and get φ_n upto $\epsilon/8^{|n|}$. Take an open interval around each n of length $\epsilon/8^{|n|}$ and modify your φ_n during these intervals so that it ends at zero at integers. Set $\varphi = \varphi_n$ on $[n.n + 1]$ and show this does.

Show that the result holds for any Radon measure. In particular, the Littlewood principle holds for any finite measure.

45. As you see, we have been more or less living on the real line. Let us start looking beyond R.

Let (X, d) be a complete separable metric space. Such spaces are also called Polish spaces. The matrix spaces (with metric inherited from the corresponding Euclidean space), $C[0, 1]$ (with supremum metric), discussed in Ex. 17 are such spaces. Of course, R, R^d, or any closed subsets of these spaces are also Polish. If you have learnt Banach spaces and Hilbert spaces, such spaces — when separable — are excellent examples of Polish spaces.

The smallest σ-field of subsets of X including its open sets is called the *Borel σ-field of X* and is denoted by $\mathcal{B}(X)$ or $\mathcal{B}_X$ or just $\mathcal{B}$. This is same as the smallest σ-field including all the open sets/ all closed sets/ all open balls/ all closed balls.

Show that $\mathcal{B}$ is countably generated, that is, $\mathcal{B} = \sigma\{A_n; n \geq 1\}$ for a sequence of sets $\{A_n, n \geq 1\}$. In particular any probability on $\mathcal{B}$ which takes only two values zero and one must be concentrated at a point.

Let μ be a finite measure on $\mathcal{B}$. Show that, just like on the real line, μ is both inner regular and outer regular. Here *inner regular* means for any Borel set B, $\mu(B) = \sup\{\mu(C) : C \subset B;\ C \text{ closed }\}$ and *outer regular* means $\mu(B) = \inf\{\mu(U) : U \supset B;\ U \text{ open }\}$.

If μ is a finite measure on $\mathcal{B}$, then just like on the real line, it is *tight* which means the following: given $\epsilon > 0$ there is a compact set K such that $\mu(K^c) < \epsilon$. Given $\epsilon > 0$, argue as follows. For an integer $n > 1$, show that there is a finite union of closed balls each of radius $1/2^n$ — denote this union by by A_n — such that $\mu(A_n^c) < \epsilon/2^n$. Look at $\cap A_n$. Separability and completeness are essential.

This is interesting. On the real line or in Euclidean spaces you can take a large closed bounded interval or box. Basically, those spaces are already σ-compact (which means countable union of compact sets) which helps. General Polish space need not be like that. In fact, an infinte dimensional Banach or Hilbert space is *never* like that!

Use tightness and inner regularity to show that for any Borel set B, $\mu(B) = \sup\{\mu(K) : K \subset B;\ K \text{ compact }\}$. This is a very useful improvement of inner regularity.

[A remark you can ignore:

Separability is important. Any set X can be made into a complete metric space by setting $d(x, y) = 0$ or 1 according as $x = y$ or $x \neq y$. Then its Borel σ-field will be its power set, simply because every subset is open. Consider the proposal: there is a set X and a probability μ on its power set such that (a) for every set A, $P(A)$ is either zero or one and (b) $P\{x\} = 0$ for every $x \in X$. Since compact sets are just finite sets in this case, tightness is clearly false for such a measure. Such a set X is unbelievably HUGE. It can not have cardinality of the real line — because then Borel σ-field of the real line would support such a measure, an impossibility. The proposal can neither be proved nor disproved.]

46. Given any two measurable functions you can talk about their maximum. Given a sequence of measurable functions also you can talk about their supremum. What if we have arbitrary collection? To understand the problem, start with an example.

    Take $(R, \mathcal{B}, \lambda)$. Fix a set $M \subset R$ which is not a Borel set. Consider the family $(I_{\{x\}} : x \in M)$. If you take the point-wise supremum then you get I_M which is not measurable.

    If you take the family $(I_{\{x\}} : x \in R)$, then for the point-wise supremum, we get the constant function $\mathbf{1}$, measurable. But what is its worth?

    In both examples my functions are all essentially (measure-theoretically) the zero function, being indicator of singleton. So the sup should be just the zero function. By blindly taking point-wise sup I ended up with a non-measurable function in the first case and the (huge) constant function one in the second case. We now let the measure play a role.

    Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $\mathbf{F}$ be a collection of real measurable functions. Say that a measurable function Z is *essential supremum* — denoted by *esup* — of the family $\mathbf{F}$ if the following holds: (i) $X \in \mathbf{F} \Rightarrow X \leq Z$ *a.e.* (Thus Z is an upper bound for all our functions.) (ii) If W is measurable; and $X \in \mathbf{F} \Rightarrow X \leq W$ *a.e.* then $Z \leq W$ *a.e.* (Thus Z is the least upper bound for our functions.) Z may take the value ∞

    The plan is to show that any family $\mathbf{F}$ indeed has *esup*. First consider the case: μ is a probability.

Step 1: Argue that there is no loss in assuming $X \in \mathbf{F}$ and $Y \in \mathbf{F}$ implies $X \vee Y \in \mathbf{F}$. Now on assume this.

Step 2: Suppose all our functions take values in $[-1, 1]$. Put $\alpha = \sup [\int X : X \in \mathbf{F}]$. Get X_n from $\mathbf{F}$ such that $\int X_n \uparrow \alpha$. Put $Z = \sup X_n$ – here pointwise supremum. Show that this does.

Step 3 : For general collection, put $\mathbf{F}' = \{\tan^{-1} X : X \in \mathbf{F}\}$. Take the principal value in $[-\pi/2, +\pi/2]$ and interpret $\tan^{-1}(\pm\infty) = \pm\pi/2$. Apply step 2 , get Z for $\mathbf{F}'$. Show $\tan Z$ does for $\mathbf{F}$.

Step 4 : Show that *esup* of a family is unique in the sense that if Z_1 and Z_2 are two such then $\mu(Z_1 \neq Z_2) = 0$.

Show that esup exists even when we have a finite measure space.

Show that esup exists in σ-finite measure spaces too.

Show that esup may not exist if the measure space is not σ-finite. Look at first example above.

[A remark which you can ignore:

Let $\mathbf{L}$ be the collection of all real measurable functions on $(\Omega, \mathcal{A}, P)$, a probability space. For fun, let us say that a subset $\mathbf{F} \subset \mathbf{L}$ is bounded above if there is $g \in \mathbf{L}$ such that $f \in \mathbf{F} \Rightarrow f \leq g \ \ a.e.[P]$. Say that h is sup of $\mathbf{F}$ if it is an upper bound and g is an upper bound implies $h \leq g \ \ a.e.[P]$, that is h is the least upper bound. Thus what we have proved above is that any bounded subset of $\mathbf{L}$ has an upper bound. This is remniscent of Real number system. From a naive point it appears that two functions may be incomparable and hence the $\leq$ a.e. order is not a linear order. But from a logical view point the statement '$(f < g) \vee (f = g) \vee (g < f)'$ has 'truth value' Ω whose probability is one. Again naively, it appears that a function need not have an inverse. But the statement '$(f = 0) \vee \exists g(fg = 1)'$ has again 'truth value' Ω having probability one. Using such interpretations Dana Scott and Robert Solovay construct models of Real number system (or of full set theory), an alternative to Cohen's Forcing. Of course, truth values are in a nice Boolean algebra. If needed one can collapse this to usual models by composing truth values with a nice homomorphism of our Boolean algebra into the two valued Boolean algebra $\{0, 1\}$.]

47. Consider the measure $\lambda \otimes \lambda$ on R^2 where λ is the Lebesgue measure on real line. Show that for any Borel set $B \subset R^2$ and any $v \in R^2$ the translate $B + v = \{u + v : u \in B\}$ is also Borel and they both have the same measure. That is, λ^2 is translation invariant. Suppose that μ is any translation invariant measure on R^2, that is, for any Borel set $B \subset R^2$ and $v \in R^2$; $\mu(B + v) = \mu(B)$. Assume that μ gives mass one for the unit square $[0, 1] \times [0, 1]$. Show that μ must be the Lebesgaue measure. (Warning: we are NOT assuming that μ is a product measure).

    λ^2 is called Lebesgue measure on R^2. Similarly, λ^n, Lebesgue measure on R^n is translation invariant.

48. Let G be a finite group and $\mathcal{A}$ to be the collection of all subsets. If μ is counting measure on G, show that it is translation invariant. It is the only such measure upto a constant multiple.

    Consider Z^n the group of n-tuples of integers equipped with the collection fo all subsets as the σ-field. Show that the counting measure is translation invariant and upto constant multiple it is the only translation invariant measure.

49. Let T^1 be the unit circle in R^2, thought of as the set of complex numbers of modulus one. Clearly under multiplication, it is a group. We equip T^1 with its Borel σ-field. Define a map $[0, 2\pi)$ to T^1 by $t \mapsto e^{2\pi i t}$, where $i = \sqrt{-1}$. In other words it is the map $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Let λ be the Lebesgue measure on $[0, 2\pi)$ and μ be the induced measure on T^1. Show that μ is translation invariant measure on T^1.

    Let T^n be the n-dimensional torus, that is, $T^1 \times \cdots \times T^1$, product of n factors. It is a group under coordinate-wise operations. Equip with product σ-field and product measure. Show that the product is translation invariant.

50. Consider R^2 with Lebesgue measure. Let $A = \{(x, y) : |x - y| \leq 1\}$. Put $f(x, y) = 0$ for points outside A. For points in A, $f(x, y)$ is ± 1 according as the point is above the diagonal or below the diagonal. Calculate the repeated integrals and 'double integral'.

51. Consider R with its Borel σ-field. Let μ_1 = Lebesgue measure and μ_2 = counting measure. Let $D = \{(x, y) : x = y;\ 0 \leq x, y \leq 1\}$ and f be its indicator function. Calculate the repeated integrals. Of course, there is no question of double integral because we have not defined product measure when one of the measures is not σ-finite.

52. Let μ be a finite measure on $(R, \mathcal{B})$ with distribution function $F(x) = \mu(-\infty, x]$. Show $\int [F(x+a) - F(x)]\, dx = a\mu(R)$.

53. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Suppose that Z is an integrable random variable. Define for every set $A \in \mathcal{A}$, $\nu(A) = \int Z.I_A\, d\mu$. Of course ν is not necessarily positive.

    Show that $\nu(\emptyset) = 0$ and ν is countably additive – that is, for any sequence of disjoint $A_n \in \mathcal{A}$ we have $\nu(\cup A_n) = \sum \nu(A_n)$. Such things are called signed measures. Let $\Omega^+ = [\omega : Z(\omega) \geq 0]$ and $\Omega^- = [\omega : Z(\omega) < 0]$. Then both these sets are in $\mathcal{A}$. Put for any $A \in \mathcal{A}$, $\nu^+(A) = \nu(A \cap \Omega^+)$ and $\nu^-(A) = \nu(A \cap \Omega^-)$.

    Show that both these are positive finite measures and $\nu = \nu^+ - \nu^-$; just like decomposition of a function into positive and negative parts. Indeed ν^+ is called the positive part of ν and ν^- is called the negative part of ν.

54. In the above exercise if we have $Z \geq 0$, then ν is a measure. We need to explain how to integrate w.r.t. this new measure ν in terms of the old measure μ. Here it is. f is ν integrable iff $Z.f$ is μ-integrable and then $\int f d\nu = \int f\ Z d\mu$; symbolically, $d\nu = Z d\mu$, also denoted $\frac{d\nu}{d\mu} = Z$. The function Z is called density of ν w.r.t. μ.

55. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space; $(\Omega', \mathcal{A}')$ be a measurable space; $T : \Omega \to \Omega'$ be a measurable transformation and μ' is the induced measure on $\mathcal{A}'$. If f is an extended-real measurable function on Ω', show that the composed map $f(T(\omega))$, denoted by foT is an extended-real measurable function on Ω and $\int f\, d\nu = \int foT\, d\mu$ in the sense, if one side is defined so is the other side and they are equal.

    The above formula for calculating integrals w.r.t ν in terms of integrals w.r.t μ is called the change of variable formula.

56. A special case of the previous exercise, Ex 49, arises when Ω is same as Ω'. Then ν is also defined on the same space as μ. A further special case arises if we moreover have a situation when this new probability ν has density w.r.t μ, that is, $d\nu = Z d\mu$, like in Ex. 48. The Jacobian formula you learnt in calculus is nothing but an explicit computation of the density in such a situation. To understand matters the best is an example.

    Let Ω be the unit interval $[0, 1]$ with its Borel σ-field $\mathcal{B}$ and μ be the Lebesgue measure. As a consequence whenever we write $d\mu(x)$ you can as well think of usual dx.

    Take $T(x) = x^2$. The induced probability be ν. Observe that for any bounded measurable function f on $[0, 1]$ the function foT is the function $f(x^2)$ on $[0, 1]$. The change of variable formula tells you that for any bounded measurable function f on $[0,1]$,

$$\int f\,d\nu = \int f(x^2)\,d\mu.$$

Let us understand ν now. Taking $Z = 1/(2\sqrt{y})$ show that for any interval $A = (a,b)$ we have $\nu(A) = \int ZI_A\,d\mu$. Hence deduce that for any Borel set A , $\nu(A) = \int ZI_A\,d\mu$. In other words the probability ν has density w.r.t μ and the density is the Z above. So remember, from Ex. 48, $\int f\,d\nu = \int f.Z\,d\mu$. Use this in our earlier equation to get

$$\int f(x^2)\,d\mu = \int f.Z\,d\mu.$$

Deciphering the notation you get

$$\int f(x^2)\,dx = \int f(y)\frac{1}{2\sqrt{y}}\,dy.$$

If you substitute $x^2 = y$ on left side, the jacobian is $dx = 1/(2\sqrt{y})dy$, and the above formula is just what you learnt in calculus. Of course, in one dimension it is usually not called Jacobian formula.

57. Show that for $a > 0$, the integral $\int\limits_0^\infty x^{a-1}e^{-x}dx$ is finite (both as a Riemann integral as well as a Lebesgue integral. Denote its value by $\Gamma(a)$ for $a \in (0,\infty)$.

    Show that the Gamma function is a continuous function. In fact it is differentiable as many times as you wish and

$$\Gamma^{(k)}(a) = \int_0^\infty x^{a-1}(\log x)^k e^{-x}dx$$

58. Try to think about the Fubini theorem for $n > 2$. There are several Fubinis! Here is one. Consider $n = 4$ and we have $(\Omega_i, \mathcal{A}_i, \mu_i)$ for $1 \le i \le 4$. We have the product space

$$\Omega = \times_1^4\,\Omega_i; \quad \mathcal{A} = \otimes_1^4\,\mathcal{A}_i; \quad \mu = \otimes_1^4\,\mu_i.$$

    Consider a measurable function $f \in L^1(\mu)$. For each $(\omega_1,\omega_4) \in \Omega_1 \times \Omega_4$ integrate f w.r.t. $(\mu_2 \otimes \mu_3)$ to get $g(\omega_1,\omega_4)$. For each $\omega_4 \in \Omega_4$, integrate g w.r.t. μ_4 to get $h(\omega_1)$. Integrate h w.r.t. μ_1. Show all this can be done and you will get $\int f d\mu$.

59. Earlier you saw that Lebesgue measure is invariant under translations. How about multiplication by scalars? Fix a positive number a and let T be the map of R^n to itself $T(v) = av$. That is if $v = (v_1,\cdots,v_n)$ then $Tv = (av_1,\cdots,av_n)$. Show that for any Borel set B, its image $T(B)$ is again Borel and $\lambda^n(TB) = a^n\lambda^n(B)$.

60. We can use induction and Fubini (by way of doing repeated integrals) to calculate the volume of the unit ball in R^n. Let $B_n = \{(x_1, \cdots, x_n) : \sum x_i^2 \leq 1\}$. and its Lebesgue measure λ^n be denoted by v_n.

Thus for $n = 1$, we have $B_1 = [-1, +1]$ and $v_1 = 2$.

For $n = 2$ we have B_2 to be the unit disc and $v_2 = \pi$.

Let $n > 2$. We need to integrate indicator of B_n w.r.t. λ^n. First ntegrate $x_3, \cdots, x_n$ to get $\left(\sqrt{1 - x_1^2 - x_2^2}\right)^{n-2} v_{n-2}$.

$$v_n = \int_{B_2} v_{n-2}(1 - x^2 - y^2)^{(n-2)/2} = \pi v_{n-2} \int_0^1 t^{(n-2)/2} dt.$$

We used polar coordinates to evaluate the double integral. Deduce

$$v_n = \pi^{n/2} \Big/ \Gamma\left(\frac{n}{2} + 1\right).$$

61. Some of you asked about infinite products. There is a problem. Consider $\Omega = [0, 2]$. Take μ to be the Lebesgue measure. Consider the infinite product space $\Omega^\infty = [0, 2]^\infty$. Each element of this space is an infinite sequence $\langle x_n : n \geq 1 \rangle$ where $0 \leq x_i \leq 2$ for all i. Let us not bother about σ-field right now. Let $A_0 = [0, 1]$ and $A_1 = (1, 2]$. Consider an infinite box with each side either A_0 or A_1. What do you think should be the measure of such a box? How many boxes have we got? If we do indeed define product measure, can it be σ-finite?

Because of these inherent problems one restricts only to infinite products where each coordinate space is probability measure. Yes, we can define product measure.

There was a recurrent theme in Exercises 47 - 49, about translation invariance. This is an important concept. Suppose you have a group G. Suppose it has a nice topology. You can then define its Borel σ-field. If the topology behaves well (?) with the group operation, you can expect the σ-field to be translation invariant. That is if B is a Borel set, then so will be $B + x$ for any $x \in G$. It makes sense to ask if there is a good (?) translation invariant measure on the Borel σ-field. For example the measure identically zero is not good. So is the measure which gives value zero for empty set and ∞ for non-empty sets. Yes, we can precisely formulate this problem. We can show that if G is locally compact (Haussdorf) group then there is such a good measure — called Haar measure. Luckily, upto a constant multiple, such a measure is unique too!

62. Here is an application to heat equation.

Let $p(t,x,y)$ be defined on $(0,\infty)\times R\times R$ as follows:

$$p(t,x,y)=\frac{1}{\sqrt{2\pi t}}\exp\{-\frac{1}{2t}(x-y)^2\}.$$

Show that for every $y\in R$ as a function of (t,x) this function satisfies the following pde, called 'heat equation'.

$$\frac{\partial p}{\partial t}=\frac{1}{2}\frac{\partial^2 p}{\partial x^2};\quad \text{for } (t,x)\in(0,\infty)\times R.$$

Let φ be a continuous function with compact support on R. Define $u(t,x)=\int\limits_R p(t,x,y)\varphi(y)dy$ for $(t,x)\in(0,\infty)\times R$ and $u(0,x)=\varphi(x)$. Thus u is defined on $[0,\infty)\times R$.

Show that u is a continuous function on $[0,\infty)\times R$ (use DCT after changing the variable y so that you have one term in the integrand as $\exp\{-u^2/2\}$) and satisfies the heat equation in $(0,\infty)\times R$ (justify differentiation under the integral sign). Let y be fixed. You need to know $(1/\sqrt{2\pi})\int\exp\{-u^2/2\}du=1$. Assume it or (if not yet bored) prove it.

[background: Here R represents an infinite rod. The function $u(t,x)$ describes the amount of heat at time t at position x on the rod; initially the heat distribution is $\varphi(y)$ at position y. The function p is called the Gaussian kernel and is the fundamental solution of the heat equation. As a function of x, $p(t,x,y)$ tells you the distribution of heat at time t; initially unit amount is put at y.]

63. Here is an application of Fubini theorem.

Let us denote by $g(t,x)$ the function $p(t,x,0)$ of the above exercise. Then the function $p(t,x,y)$ of the above exercise is nothing but $g(t,x-y)$ and the function $u(t,x)$ is nothing but $\int g(t,x-y)\varphi(y)dy$. Such a thing is called convolution. Here is precise definition.

Consider $L^1(R,\mathcal{B},\lambda)$. For f and g in L^1 define $h(x)=\int f(x-y)g(y)d\lambda(y)$. Show that this integral is finite for a.e. y. This function is denote by $f*g$ and is called convolution of the functions f and g (for those y for which the integral does notd exist, take it zero or your favourite number). Show that this function is again in L^1. Thus you can think convolution as a 'multiplication' in L^1. Note in passing that usual pointwise product of two functions in L^1 need not be in L^1.

Show that convolution is commutative, that is, $f*g = g*f$; and associative, that is $f*(g*h) = (f*g)*h$; and distributive, that is $f*(g+h) = f*g+f*h$; for any three L^1 functions f, g, h.

Convolution has the following nice interpretation. Assume that f and g are non-negative functions. Thus you can think of two finite measures $d\mu = fd\lambda$ and $d\nu = gd\lambda$ on R. Consider the product measure $\mu \otimes \nu$ on R^2. Consider the map $T : (x, y) \mapsto x + y$ on R^2 to R. Let the induced measure on R by this map T — when R^2 is equipped with $\mu \otimes \nu$ — be denoted by α. Then α is a measure on R and $d\alpha = f * gd\lambda$. Prove this. Remember measures can be handled by distribution functions.

You will see more about convolution later.

64. You saw two principles of Littlewood: every Borel set is nearly an open set; every Borel function is nearly a continuous function. Here is the third principle: every convergence is nearly uniform convergence.

    Precise statement is as follows. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Suppose $f_n \to f$ a.e., all functions are assumed measurable. Let $\epsilon > 0$ be given. Then there is a set $A \in \mathcal{A}$ such that f_n converges uniformly to f on the set A and $\mu(A^c) < \epsilon$. This is proved as follows. For integers $n, k \geq 1$, put

    $$A_{n,k} = \{\omega : |f_i(\omega) - f(\omega)| < 2^{-k} \quad \text{for all} \;\; i \geq n\}$$

    If you fix k then these sets increase to Ω alomst surely. Fix a large integer n_k so that the corresponding set leaves out only a set of measure at most $\epsilon/2^{-k}$. Take intersection of these sets so obtained.

    This result is called Egoroff's theorem. If μ is not finite this may be false. More precisely, on any σ-finite non-finite measure space there is always a sequence of functions which converges a.e. but fails to converge uniformly on any set A with $\mu(A^c) < \infty$.

65. Do you remember matrices? Here are matrices with uncountably many rows and columns, called Kernels. Consider finite measure space $(\Omega, \mathcal{A}, \mu)$. Let $K \in L^2(\Omega \times \Omega, \mu \otimes \mu)$. For $f \in L^2(\mu)$ put

    $$(Kf)(x) = \int K(x, y)f(y)d\mu(y).$$

    Show that this integral exists (almost everywhere) and defines again a function Kf. Show that this function Kf is again in $L^2(\mu)$. Show that this is a linear map. Show that $||Kf|| \leq ||K|| \times ||f||$. Here norms refer to the appropriate spaces where the functions are living.

    Thus R^n is replaced by $L^2(\mu)$. Vector $v = (v_i; 1 \leq i \leq n)$ is replaced by function f. Matrix (a_{ij}) is replaced by Kernel $K(x, y)$. Finally summation over j in $(Av)(i) = \sum_j a_{ij}v_j$ is replaced by integration over y to get

$\int K(x,y)f(y)d\mu(y)$. In fact the matrix set up is a special case if you take $\Omega = \{1, 2 \cdots, n\}$ and μ to be counting measure. Just remember elements of R^n are just functions defined on the set $\{1, 2, \cdots, n\}$ and $n \times n$ matrix is just a function defined on the product space $\{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$.

66. The notion of convergence behaves well with usual operations of addition etc. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. All functions below are real valued and measurable.

    (a) Let $f_n \to f$ a.e. and $g_n \to g$ a.e.

    Show $f_n + g_n \to f + g$ a.e.; $f_n g_n \to fg$ a.e.; for $c \in R$, $cf_n \to cf$ a.e.

    If φ is a continuous function on R to R show that $\varphi o f_n \to \varphi o f$.

    (b). Let $f_n \to f$ and $g_n \to g$ in measure. Show that $f_n + g_n \to f + g$ in measure; for any real number c, $cf_n \to cf$ in measure.

    If φ is a uniformly continuous function on R to R, then $\varphi o f_n \to \varphi o f$ in measure. This need not be true if φ is only continuous. Try $(R, \mathcal{B}, \lambda)$ and $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$ and $\varphi(x) = x^2$. Do you think $f_n g_n \to fg$ in measure?

    However if μ is finite, then show that $\varphi o f_n \to \varphi o f$ in measure whenever $f_n \to f$ in measure and φ is continuous function on R to R. Try to get a bounded interval $[-K, +K]$ such that all the values of f_n and f lie here with a large measure and proceed.

    (c) Show that addition and scalar multiplication are well behaved for L^p convergence also.

67. Suppose you have a finite measure space $(\Omega, \mathcal{A}, \mu)$. Show that for $1 \leq r \leq s$, if $f \in L^s$, then $f \in L^r$. Thus L^p spaces decrease as p becomes large. Try integrating $|f|^r$ on the sets where it is larger than one and smaller than one separately.

    Consider $(R, \mathcal{B}, \lambda)$. Give examples of functions which are (a) in L^1 but not in L^2 (b) in L^2 but not in L^1 (c) in none of them (d) in both of them. If it helps, you may think on the set of natural numbers with counting measure instead of R.

68. Given a measure space, we defined the L^p spaces for $1 \leq p < \infty$. We can define the space L^∞ also.

    Consider the function f on R defined as follows. $f(x) = x$ for x irrational in $[0, 1]$; $f(x) = 88$ for x rational in $[0, 1]$ and $f(x) = 0$ for $x \notin [0, 1]$. what is the 'supremum' of the function? it appears to be 88. Such an answer forgets the measure and blindly calculates the pointwise sup. Note that f assumes the value 88 only on a set of measure zero. f is smaller than one on a set of full measure. So from a measure theoretic point of view supremum should be one. You can not make it smaller than one, because if you take any number $a < 1$, then on a set of positive measure f takes values larger than a. This idea is made precise below.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space — μ is not identically zero. Let f be a non-negative real measurable function. Say $a \in [0, \infty]$ is an 'essential upper bound' for f if $\mu(f > a) = 0$. Show that there is a smallest 'essential upper bound' for f. That is, there is $b \in [0, \infty]$ such that it is an eub for f; and whenever a is an eub then $b \leq a$. This b is called the essential supremum of f. For example if we consider $(R, \mathcal{B}, \lambda)$ and $f(x) = x^+$, then the esup is ∞. For the example of the earlier para, the esup is one.

$L^\infty(\Omega, \mathcal{A}, \mu)$ is the collection of all real measurable functions such that esup of $|f|$ is finite. For functions in this space define $||f||_\infty$ as the esup of $|f|$.

Show that $||f||$ is also the supremum of all numbers c satisfying the condition $\mu\{|f| > c\} > 0$.

Show that the $||\cdot||$ is a norm on the space L^∞. This means (1) $||f+g|| \leq ||f|| + ||g||$; (2) $||cf|| = |c| \times ||f||$ for $c \in R$; $||f|| = 0$ iff f is essentially the zero function, that is, $\mu(f \neq 0) = 0$.

Show that the space is complete, that is, every Cauchy sequence converges.

69. Consider $L^p(R)$ where $1 \leq p < \infty$. Show that continuous functions with compact support are dense.

    Show that C^∞ functions with compact support are dense.

    Show that the space L^p is separable, that is, it has a countable dense set.

    Show that the space L^p is translation invariant. This means the following. For any function $f : R \to R$ and any real number a define translate of f by a to be the function $f_a(x) = f(x+a)$. Show that if $f \in L^p$ and $a \in R$, then $f_a \in L^p$.

    Show that the L^p distance is translation invariant. That is, for $f, g \in L^p$ and $a \in R$; $||f - g|| = ||f_a - g_a||$.

    Fix $f \in L^p$. Show that the map $a \mapsto f_a$ from R to L^p is uniformly continuous. First show that this is correct when f is continuous function with compact support. Approximate general functions.

70. Consider $\Omega = R^2 - \{(0,0)\}$ with Lebesgue measure λ.

    Given $(x,y) \in \Omega$ there is a unique pair of numbers (r,θ) such that (i) $r > 0$ (ii) $0 \le \theta < 2\pi$ and (iii) $x = r\cos\theta; y = r\sin\theta$. Let $\Omega' = \{(r,\theta) : r > 0; 0 \le \theta < 2\pi\}$. Let T be the map that associates (r,θ) described above with (x,y). Then T is a one-to-one map on Ω onto Ω'.

    Note that Ω' is nothing but $(0,\infty) \times [0,2\pi)$. Equip this with product of Borel σ-fields on the two coordinate spaces. Then T is a measurable map.

    Equip $(0,\infty)$ with the measure $r\, dr$, that is measure of any interval (a,b) is $\int_a^b r dr = (b^2 - a^2)/2$. Equip $[0,2\pi)$ with Lebesgue measure. Equip the product space Ω' with the product measure μ, say. Then the measure induced by T on Ω' is μ.

    We have proved this in class but verify all the details. This and the next one are trivial special cases of 'Jacobian theorem' or 'change of variable formula for double integrals'.

71. Consider R^2 with Lebesgue measure λ. Fix a non-singular 2×2 matrix A. Denote also by A the map from R^2 to R^2 given by $v \mapsto Av$. Show that this is one-to-one, measurable, and onto R^2.

    What is the measure μ induced by this transformation? Let c denote absolute value of the determinant of A. Since A is non-singular, $c \neq 0$. The claim is $\mu = c\lambda$. To see this first show that μ is translation invariant and finite for bounded sets. Conclude that μ is a multiple of λ. the only question is: what is that constant?

    We need one test case. For example we can look at the image of the square $[0,1] \times [0,1]$ and calculate its measure. Do this in simple steps. Suppose A is diagonal matrix; A is interchange of coordinates; A sends (x,y) to $(x+y,y)$. Then try general case.

    This is true for R^n also. That is, if we take a non-singular $n \times n$ matrix A and consider the map $v \mapsto Av$ on R^n then the induced measure is $c\lambda$ where c is the absolute value of the determinant of A. But you need not do this. We shall use this for $n = 4$ later.

72. Returning to an earlir theme of Haar measures, this and next few exercises provide some more examples. Just as much of the analysis on R is done with Lebesgue measure, so is it on (nice) groups: one uses the Haar measure. It is important to know 'the' Haar measure. Instead of prescribing measures of Borel sets, we specify integrals (of continuous functions with compact support). Of course whether prescription of integral leads to a measure is an issue, but we do not need to tackle this. From the integrals we prescribe, the measure would be clear.

    Consider the group $G = (0, \infty)$ under multiplication. The Haar integral is given by
$$\int f d\mu = \int \frac{f(x)}{x} dx.$$
    In other words the Haar measure μ is absolutely continuous w.r.t. the usual Lebesgue measure and the density is given by $1/x$. That is $d\mu = dx/x$.

    Equivalently, measure of any interval $[a, b] \subset (0, \infty)$ is given by
$$\mu(a, b) = \int_a^b \frac{1}{x} dx = \log b - \log a.$$

    Equivalently, it is the measure induced by the Radon distribution function $F(x) = \log x$ for $x \in (0, \infty)$.

    Equivalently consider the map from R to $(0, \infty)$ given by $Tx = \exp\{x\}$, then μ is the measure induced by T when R has Lebesgue measure.

73. Let G be the set of non-zero real numbers under multiplication. Then
$$d\mu = \frac{1}{|x|}\ dx \quad (i.e.) \quad \mu(a, b) = \int\limits_a^b \frac{1}{|x|} dx$$
    for any interval $[a, b] \subset G$,

74. Consider G to be the set of 2×2 non-singular matrices, regarded as a subset of R^4. More specifically (x_1, x_2, x_3, x_4) corresponds to the matrix whose first row is (x_1, x_2) and second row (x_3, x_4). Of course G does not correspond to all of R^4 but only an open subset of R^4. The Haar integral is given by
$$\int f d\mu = \int \frac{f(X)}{|det(X)|^2} dX; \quad dX = dx_1 dx_2 dx_3 dx_4$$

It is interesting to realize that the group is noncommutative and hence there are two definitions of 'invariance'. Say that μ is left invariant if for every Borel set B and every $g \in G$; $\mu(gB) = \mu(B)$. Say that μ is right invariant if for every Borel set B and $g \in G$; $\mu(Bg) = \mu(B)$. Here $gB = \{gx : x \in B\}$ and $Bg = \{xg : x \in B\}$. These are Borel sets because on all of G, the map $x \mapsto xg$ and $x \mapsto gx$ are homeomorphisms. The measure above is both left and right invariant. Use exercise 71.

75. Consider the group G of 2×2 non-singular matrices with second row $(0, 1)$. Thus such a matrix is determined by its first row (x, y) — where x must be non-zero due to non-singularity. Thus G can be identified with $R^2 - \{(x, y) : x = 0\}$.

$$\int f d\mu = \int \frac{f(X)}{|det(X)|^2} dX; \quad dX = dxdy.$$

$$\int f d\nu = \int \frac{f(X)}{|det(X)|} dX; \quad dX = dxdy.$$

Then μ is left invariant measure and ν is right invariant. Unlike in the previous example (which is also a noncommutative group), here the left and right Haar measures are different.

76. Let G be the group of 2×2 non-singular matrices of the form

$$\begin{pmatrix} x & y \\ 0 & \frac{1}{x} \end{pmatrix}$$

Again this can be identified with $R^2 - \{(x, y) : x = 0\}$.

$$\int f d\mu = \int \frac{f(x, y)}{x^2} dxdy; \quad \int f d\nu = \int f(x, y) dxdy.$$

The measure μ is left Haar measure and ν is right Haar measure. Thus right Haar measure is just the Lebesgue measure.

77. Let G be the set of non-zero complex numbers under multiplication. We think of G as $R^2 - \{(0, 0)\}$. Of course this is abelian. Haar integral is given by

$$\int f d\mu = \int \frac{f(x, y)}{x^2 + y^2} dxdy.$$

Of course, there are more interesting groups, like the unitary matrix group, orthogonal matrix group etc; we do not discuss.

78. Let μ be a σ-finite measure on $(R, \mathcal{B})$.

    μ is said to be discrete if there is a countable set S such that $\mu(S^c) = 0$, thus μ puts its mass on a countable set.

    μ is said to be continuous if for all x, $\mu\{x\} = 0$. Thus for example if μ is a Radon measure then this is same as saying that its distribution function is a continuous function.

    μ is said to be absolutely continuous if $\mu << \lambda$, that is absolutely continuous w.r.t. the Lebesgue measure.

    μ is said to be singular if $\mu \perp \lambda$, that is singular w.r.t. the Lebasgue measure.

    The only measure which is common to any two of the three classes — discrete, continuous singular, absolutely continuous — is the zero measure.

    If μ is a σ-finte measure on R, show that there are three uniquely defined measures μ_1, μ_2, μ_3 such that (i) $\mu = \mu_1 + \mu_2 + \mu_3$ (ii) μ_1 is discrete, μ_2 is continuous singular; μ_3 is absolutely continuous.

79. Minkowski says that L^p norm of a sum of functions is at most the sum of their L^p norms. Here is a continuous version. Prove it.

    Let $f(x, y)$ be a non-negative measurable function of two variables. Then

$$\left[\int \left| \int f(x,y)dy \right|^p dx\right]^{1/p} \leq \int \left(\int |f(x,y)|^p dx \right)^{1/p} dy.$$

80. Holder says that $||fg||_1 \leq ||f||_p ||g||_q$ when $(1/p) + (1/q) = 1$ (always $1 < p, q < \infty$). Here is a generalization or self improvement.

$$||f_1 f_2 \cdots f_k||_r \leq ||f_1||_{p_1} ||f_2||_{p_2} \cdots ||f_k||_{p_k} \quad \text{if} \quad \sum_1^k \frac{1}{p_i} = \frac{1}{r}.$$

81. Consider the coin tossing space $\Omega = \{0, 1\}^\infty$ and the product $(1/2, 1/2)$ measure on its σ-field $\mathcal{B}$. It is the unique measure μ such that $\mu(A_s) = 1/2^{|s|}$. Recall, from exercise 8, that Ω consists of all infinite sequences of zeros and ones. For a finite sequence s of zeros and ones, $A_s \subset \Omega$ consists of all those infinite sequences which start with s. And $|s|$ denotes length of s, length of empty sequence is zero.

    There is a measure μ as decsribed above because, the prescription is finitely additive and hence (!) countably additive on the semi-field of sets $\mathcal{S}$.

Recall that $\mathcal{S}$ consists of emptyset and the collection of all sets A_s as s varies over all finite sequences of zeros and ones. When s is the empty sequence, we take $A_s = \Omega$.

Define a map $T : \Omega \to [0, 1]$ by

$$T(\epsilon) = \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}; \quad \text{where} \quad \epsilon = (\epsilon_1, \epsilon_2, \cdots).$$

Show that the induced measure on $[0, 1]$ is the Lebesgue measure (compute induced measure of the diadic intervals and argue).

If $T(\epsilon) = T(\eta)$ then show that there is an n such that $\epsilon_i = \eta_i$ for $i < n$; one of ϵ_n , η_n is zero and the other is one. If $\epsilon_n = 1$ (and consequently, $\eta_n = 0$), then $\epsilon_i = 0$ for all $i > n$ while $\eta_i = 1$ for all $i > n$. Show this.

Use the above observation to show the following: T is one to one if we remove a countable set from Ω. For example, let us remove all points which have only ones after some stage and have at least one zero. Thus the point all of whose coordinates are one is not removed. Let Ω_0 be the remaining set and$\mathcal{B}_0$ be its σ-field — this means $\mathcal{B}_0 = \{B \in \mathcal{B} : B \subset \Omega_0\}$. Show that for any set $B \in \mathcal{B}_0$ its image, namely, $T(B)$ is a Borel seet of $[0, 1]$.

We have an interesting situation. We have two spaces $(\Omega_0, \mathcal{B}_0, \mu)$ and $([0, 1], \mathcal{B}, \lambda)$ and T associates with each point of $\omega \in \Omega_0$ a point $T(\omega) \in [0, 1]$. Firstly, T is one-one and onto. Secondly, T preserves Borel sets, which means the following. For any $B \in \mathcal{B}_0$ we have $T(B) \in \mathcal{B}$ and for any set $A \in \mathcal{B}$ we have $T^{-1}(A) \in \mathcal{B}_0$. Finally T preserves the measures which means the following. $\mu(B) = \lambda(T(B))$ for $B \in \mathcal{B}_0$ and $\lambda(A) = \mu(T^{-1}(A))$ for $A \in \mathcal{B}$.

Thus if you rename points of Ω_0 with the help of T, you get $[0, 1]$. This renaming preserves the entire structure: σ-fields as well as measures. In other words it is an *isomorphism* of the measure spaces.

82. Consider the coin tossing space (Ω, μ) of the above exercise. It admits several symmetries. Let $T : \Omega \to \Omega$ be defined by

$$T(\epsilon_1, \epsilon_2, \cdots) = (1 - \epsilon_1, 1 - \epsilon_2, \cdots).$$

Thus T flips each coordinate: makes zero to one and one to zero.Show that this is measure preserving. That is, $\mu(T^{-1}B) = \mu(B)$ for any $B \in \mathcal{B}$.

In fact you need not flip all coordinates, you fix some coordinate indices, and flip only those places for all points of Ω. In other words let $S \subset N$ and put

$$T(\epsilon_1, \epsilon_2, \cdots) = (\eta_1, \eta_2, \cdots); \quad \eta_i = 1 - \epsilon_i \text{ for } i \in S; \; \eta_i = \epsilon_i \text{ for } i \notin S.$$

Fix any permutation π of N, that is, a one-to-one map of N onto itself. Define a map on Ω to itself, to be again denoted by the same symbol π, by

$$\pi(\epsilon_1, \epsilon_2, \cdots) = (\epsilon_{\pi(1)}, \epsilon_{\pi(2)}, \cdots).$$

Thus action of π on any point is to permute its coordinates. Then π is also measure preserving map of Ω.

You should realize that all these maps translate to the unit interval too, in view of the earlier exercise. But you need not translate.

83. We shall continue the story of Cantor set in this and next exercise.

    We have seen that Lebesgue measure is translation invariant, thus for Borel A and real b; $\lambda(A + b) = \lambda(A)$. If A, B are compact sets then $A + B = \{x + y : x \in A; y \in B\}$ is again compact. Is there any relation between $\lambda(A + B)$ and $\lambda(A), \lambda(B)$? Not really.

    Show that Cantor set C has Lebesgue measure zero, but $C + C = [0, 2]$. Show this by remembering the construction of Cantor set and arguing: if $K_n \downarrow K$ compact sets and if for each n, $K_n + K_n = [0, 2]$ then we must have $K + K = [0, 2]$ also. Show that there is a Borel set N of Lebesgue measure zero such that $N + N = R$.

84. This and the next exercise depend on Zorn's lemma, which is a statement equivalent to the axiom of choice. If you have not heard of this Lemma earlier, you need not work out these two exercises. Read and understand them so that you do not miss the fun.

    Here is Zorn's Lemma. Let $(P, \leq)$ be a partially ordered set. Say that a subset $C \subset P$ is a chain if elements of C are comparable, that is, $x, y \in C$ implies that either $x \leq y$ or $y \leq x$. An upper bound for a chain C is an element $u \in P$ such that for all $x \in C$ we have $x \leq u$. A maximal element in P is an element $m \in P$ such that $x \in P; m \leq x \Rightarrow x = m$. Thus there is nothing bigger than m, of course there may be things not comparable to m. Zorn's lemma says the following: if you have a poset in which every chain has an upper bound, then the poset has at least one maximal element.

    Recall that Hamel basis for R is a basis of R regarded as a vector space over the field of rationals Q. Equivalently, it is a set B such that the following two condittions hold. (1) (linear independence) If $(x_i : 1 \leq i \leq k)$ are distinct elements of B and $(r_i : 1 \leq i \leq k)$ are rationals and $\sum r_i x_i = 0$ then each r_i is zero. (2) (generates R) Given any $x \in R$, there are finitely many rationals (r_i) and elements of B, say, (x_i) such that $x = \sum r_i x_i$.

    Argue that a Hamel Basis exists, assuming Zorn's lemma. Let B be one such. Using an old observation, show that if B is measurable, then B can not have positive measure (if it had, then every real is a rational linear combination of at most two elements of B).

Use previous exercise to argue that there is a Hamel basis B contained in the Cantor set and hence it is of Lebesgue measure zero. Let $A = \cup_{r \in Q} rB$. Here $rB = \{rx : x \in B\}$. Show that A has Lebsgue measure zero. Set $A_1 = A$; $A_2 = A_1 + A_1$ and in general $A_{n+1} = A_n + A_n$. Show that if A_n is Lebesgue measurable then it should be null (otherwise, argue every real is a rational linear combination of at most $2n$ basis elements!). Show that the union of all the A_n equals R. Deduce that there is a Lebesgue null set N such that $N + N$ is not Lebesgue measurable.

85. Let $N = \{1, 2, \cdots\}$, the set of natural numbers. A collection $\mathcal{U}$ of subsets N is called a filter if it is a proper subcollection of power set of N which is closed under super sets and finite intersections. That is, (i) $A \in \mathcal{U}$; $B \supset A \Rightarrow B \in \mathcal{U}$ and (ii) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$ and (iii) $\emptyset \notin \mathcal{U}$.

For example $\mathcal{U}_1 = \{A : A^c \text{ is finite}\}$; $\mathcal{U}_2 = \{A : 52 \in A\}$; $\mathcal{U}_3 = \{A : A \text{ includes all primes}\}$; are all filters. It is easy to give examples which are not filters.

A filter $\mathcal{U}$ is called an ultrafilter if it is not properly contained in any filter. That is if $\mathcal{V}$ is a filter which includes $\mathcal{U}$ then $\mathcal{V} = \mathcal{U}$. Show that a filter is an utrafilter iff for any $A \subset N$, either A or A^c is in the filter. Thus an ultrafilter is a choice of one element from each pair $\{A, A^c\}$ so that thee chosen sets satisfy the conditions of filter.

For example $\mathcal{U}_2$ above is an ultrafilter. Say that an utrafilter is free ultrafilter if it includes all co-finite sets. It is not easy to give examples of free ultrafilter. Use Zorn's lemma to show that every filter can be extended to an ultrafilter. In particular if you extend $\mathcal{U}_1$ above you get free ultrafilter.

Suppose that $(x_n : n \in N)$ is a bounded sequence of real numbers. Of course, in general such a sequence need not converge. We now make a definition. Say that the sequence converges to number x *along the ultrafilter* $\mathcal{U}$ if the following holds: given any $\epsilon > 0$, there exists $S \in \mathcal{U}$ such that $\{x_i; i \in S\} \subset (x - \epsilon, x + \epsilon)$.

If this definition looks abstract, imagine for a moment that I do not have ultrafilter, but the filter $\mathcal{V}$ of cofinite sets. Show the following: The sequence x_n converges to x in the usual analysis sense if and only if the following holds: given any $\epsilon > 0$, there exists $S \in \mathcal{V}$ such that $\{x_i; i \in S\} \subset (x - \epsilon, x + \epsilon)$.

Continuing with a bounded sequence (x_n) of real numbers, show that the set

$$\bigcap_{S \in \mathcal{U}} \overline{\{x_n : n \in S\}}$$

consists of exactly one real number, denote it by x. Here $\overline{A}$ denotes the closure of the set A. Show that x_n converges to x along the ultrafilter $\mathcal{U}$.

Thus ultrafilter directs **every** bounded sequence to converge!

86. Consider R^∞. Recall,

$$\mathcal{B}^\infty = \sigma\{B_1 \times B_2 \times \cdots : B_i \subset R, \text{ Borel, for all } i\}.$$

Show that

$$\mathcal{B}^\infty = \sigma\{B_1 \times B_2 \times \cdots : B_i \subset R, \text{ Borel for all } i \text{ and } (\exists n)(\forall i \geq n) B_i = R\}$$

$$= \sigma\{B_1 \times B_2 \times \cdots : B_i \subset R, \text{ Borel for all } i \text{ and } (\exists n)(\forall i \neq n) B_i = R\}$$

$$= \sigma\{B_1 \times B_2 \times \cdots : B_i \subset R, \text{ an interval for all } i \text{ and } (\exists n)(\forall i \geq n) B_i = R\}$$

$$= \sigma\{B_1 \times B_2 \times \cdots : B_i \subset R, \text{ an interval for all } i \text{ and } (\exists n)(\forall i \neq n) B_i = R\}.$$

For a fixed n, the set in the first braces is a box based on the first n coordinates; the set in the second braces is a box based only on the n-th coordinate; the set in the third braces is box with interval sides based on the first n coordinates; the set in the last braces is a box with interval side based only on the n-th coordinate.

The above equalities are more than mere technical observations. You should keep an eye on the complexity of the set. The boxes in the definition of $\mathcal{B}^\infty$ are truly infinite dimensional boxes, that is, to decide whether a point is in your box or not you need to check all its coordinates. The sets described later are finite dimensional boxes, that is, to decide whether a point is in your box or not you need to check only finitely many coordinates. Sometimes it is enough to check only one coordinate!

Here are some examples of sets in $\mathcal{B}^\infty$.

$$A = \{(x_1, x_2, \cdots) : |x_m - x_n| < 0.01; \quad (\forall m, n \geq 255)\}.$$

$$B = \{(x_1, x_2, \cdots) : \{x_n\} \quad \text{is a cauchy sequence}\}.$$

$$C = \{(x_1, x_2, \cdots) : x_1 < x_2 < x_3 \cdots\}.$$

$$D = \{(x_1, x_2, \cdots) : x_1 = x_2 = x_3 = \cdots\}.$$

$$E = \{(x_1, x_2, \cdots) : \{x_n\} \quad \text{is a monotone sequence}\}.$$

$$F = \{(x_1, x_2, \cdots) : \text{no two coordinates are equal}\}.$$

$$G = \{(x_1, x_2, \cdots) : \sum x_n \text{converges}\}.$$

$$H = \{(x_1, x_2, \cdots) : \frac{1}{n} \sum x_n \to 51\}.$$

$$J = \{(x_1, x_2, \cdots) : \text{infinitely many coordinates are integers}\}.$$

$$K = \{(x_1, x_2, \cdots) : 31 < \sum_1^n x_i < 58 \text{ for infinitely many values of } n\}.$$

Observe that none of the sets above are finite dimensional sets. That is, to decide whether a point, $(x_1, x_2, \cdots) \in R^\infty$, is in your set or not you need to check all its coordinates.

Here is a more complicated set in $\mathcal{B}^\infty$. Let $r_1, r_2, \cdots$ be an enumeration of all rational numbers in $[0, 1]$.

$$S = \{(x_1, x_2, \cdots) : (\exists f \ \text{ continuous function on } [0, 1]) \ (\forall i) \ f(r_i) = x_i\}.$$

This is not just set theory, you need to use facts from analysis.

Here is another set. Now let $r_1, r_2, \cdots$ be an enumeration of all rational numbers in R.

$$T = \{(x_1, x_2, \cdots) : (\exists f \ \text{ continuous function on } R) \ (\forall i) \ f(r_i) = x_i\}.$$

87. Here are some examples of extended real measurable functions on $(R^\infty, \mathcal{B}^\infty)$.

$$f(x_1, x_2, \cdots) = \limsup x_n; \quad g(x_1, x_2, \cdots) = \liminf x_n$$

Take any measurable function $f : R^{100} \to R$ and put

$$g(x_1, x_2, \cdots) = f(x_1, \cdots, x_{100}).$$

$$h^*(x_1, x_2, \cdots) = \limsup \sum_{i=1}^n x_i; \quad h_*(x_1, x_2, \cdots) = \liminf \sum_{i=1}^n x_i.$$

Suppose you have a collection $\mathbf{F}$ of real measurable functions on R^∞. Suppose that $\mathbf{F}$ is a vector space closed under increasing pointwise limits (that is, each f_n is in your collection and $f_n \uparrow f$ and f real valued implies f is in your collection). If $\mathbf{F}$ includes indicators of boxes used in the definition of $\mathcal{B}^\infty$ then $\mathbf{F}$ equals the collection of all real measurable functions on R^∞.

88. We have seen some interesting sets in $\mathcal{B}^\infty$ and some measurable functions in the earlier exercise. Here are some interesting facts about integration. So let us denote by λ the product measure $\mu_1 \otimes \mu_2 \otimes \cdots$ on R^∞ where each μ_i is a probability on R. Suppose that f_i is a measurable; bounded or non-negative; function on R for each i. Fix an $n \geq 1$ and define the function f on R^∞ by

$$f(x_1, x_2, \cdots) = f_1(x_1) f_2(x_2) \cdots f_n(x_n).$$

Show that

$$\int f d\lambda = \prod_{i=1}^n \left(\int f_i d\mu_i \right).$$

89. There is a simple but powerful and useful result regarding measures of some complicated sets. Use the notation λ for the product measure as in the earlier exercise. Let us define some σ-fields on R^∞ which are contained in $\mathcal{B}^\infty$. For each $n \geq 1$, let us put

$$\mathcal{F}^n = \sigma\{B_1 \times B_2 \times \cdots : B_i \subset R \text{ Borel for each } i; B_j = R \text{ for } j \leq n\}.$$

Do not get confused. Earlier we had, in class, the σ-fields $\mathcal{F}_n$; these are sets that *depend only* on the first n coordinates. Now we have $\mathcal{F}^n$; these are sets that *do not depend* on the first n coordinates. To get a feel here are some sets.

$$\{(x_1, x_2, \cdots) : x_1 + x_2 + \cdots + x_{100} < 55\} \in \mathcal{F}_{100}.$$

$$\{(x_1, x_2, \cdots) : x_{101} + x_{102} + \cdots + x_{1000} < 55\} \in \mathcal{F}^{100}.$$

$$\left\{(x_1, x_2, \cdots) : \sum x_i \quad \text{converges}\right\} \in \mathcal{F}^n. \quad \text{for all } \ n.$$

$$\left\{(x_1, x_2, \cdots) : \frac{1}{n}\sum_1^n x_i \ \text{ converges}\right\} \in \mathcal{F}^n \quad \text{for all } \quad n.$$

$$\left\{(x_1, x_2, \cdots) : \frac{1}{n}\sum_1^n x_i \to 23\right\} \in \mathcal{F}^n \quad \text{for all } \quad n.$$

Put $\mathcal{T} = \cap \mathcal{F}^n$, sets that belong to all the $\mathcal{F}^n$. Clearly this $\mathcal{T}$ is a σ-field and contains several sets as seen above.

Here then is the result: If $A \in \mathcal{T}$ then either $\lambda(A) = 1$ or $\lambda(A) = 0$. Since λ is a probability this statement is same as saying either $\lambda(A) = 1$ or $\lambda(A^c) = 1$. In other words the probability λ restricted to $\mathcal{T}$ takes only two values. This is known as Kolmogorov's zero-one law. The σ-field $\mathcal{T}$ is called *tail* σ-field and sets which belong to $\mathcal{T}$ are called tail sets.

This has, for example, the following implication: either for *almost every* sequence $(x_1, x_2, \cdots)$ their sum converges or for *almost no* sequence the sum converges. In other words, points in the space R^∞ exhibit a typical behaviour, as far as convergence of their sum is concerned. Of course, almost every is w.r.t. the product measure λ and the beauty is that choice of the μ_i' s is at your dsiposal!

Proof is simple. Fix n and show that

$$A \in \mathcal{F}^n; \ \ B \in \mathcal{F}_n \Rightarrow \lambda(A \cap B) = \lambda(A)\lambda(B).$$

Deduce

$$A \in \mathcal{F}^n \ \text{ for all } n; \ \ B \in \mathcal{F}_n \ \text{ for some } n \Rightarrow \lambda(A \cap B) = \lambda(A)\lambda(B).$$

Observing $\cup \mathcal{F}_n$ is a field that generates $\mathcal{B}^\infty$, show that

$$A \in \mathcal{T}; \;\; B \in \mathcal{B}^\infty \Rightarrow \lambda(A \cap B) = \lambda(A)\lambda(B).$$

Now use the fact $\mathcal{T} \subset \mathcal{B}^\infty$ to derive the zero-one law.

90. Let us return to Real line, Borel σ-field and lebesgue measure λ. At some stage in our life (exercise 35) we showed that C^∞ functions are dense in $L^1(\lambda)$. This can be used to prove the following.

    Riemann-Lebesgue Lemma: For $f \in L^1(R)$, $\lim\limits_{n\to\infty} \int f(x) \sin nx \; dx = 0$.

    First take a C^1 function with compact support and show this using integration by parts and then proceed.

    Show that for $f \in L^1(R)$, $\lim\limits_{n\to\infty} \int f(x) \cos nx dx = 0$.

    Show that for $f \in L^1(R)$, $\lim\limits_{t\to\infty} \int f(x) \sin tx dx = 0$.

91. If $f \in L^1(R)$, its Fourier transform $\hat{f}$ is the complex valued function defined on the real line by the formula $\hat{f}(t) = \int f(x)e^{itx}dx$. Show that the integral exists and is a continuous function on R.

    If f is real valued, show that $\hat{f}(-t) = \overline{\hat{f}(t)}$. Here the overline denotes complex conjugate. What if f is not real valued?

    Assume that f is real valued. Show that $\hat{f}$ is positive definite, that is, for complex numbers $c_1, c_2, \cdots, c_n$ and for real numbers $t_1, t_2, \cdots, t_n$; $\sum\limits_{k,l} c_k \overline{c_l} \hat{f}(t_k - t_l) \geq 0$. This is same as saying that the $n \times n$ matrix whose (k,l)-th entry is $\hat{f}(t_k - t_l)$ is positive definite.

    Show that $\hat{f}(t) \to 0$ as $t \to \infty$.

    If $xf \in L^1$ show that $\hat{f}$ is differentiable and its derivative is given by $(\hat{f})'(t) = i \int xf(x)e^{itx}dx$.

92. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Say that a set $A \in \mathcal{A}$ is a μ-atom if (i) $\mu(A) > 0$ and (ii) $B \subset A$, $B \in \mathcal{A}$ implies $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

    Show that μ can have at most countably many disjoint atoms.

    Say that μ is atomic if either it is the zero measure or there are countably many disjoint μ-atoms $\{A_n : n \geq 1\}$ such that $\mu(\Omega - \cup A_n) = 0$. That is, the whole space Ω is essentially union of atoms. Say that μ is non-atomic if it has no atoms. That is, $A \in \mathcal{A}$, $\mu(A) > 0$ implies there exists $B \subset A$, $B \in \mathcal{A}$ with $0 < \mu(B) < \mu(A)$.

    Show that there are two uniquely determined σ-finite measures μ_1 and μ_2 such that (i) μ_1 is atomic, μ_2 is non-atomic and (ii) $\mu = \mu_1 + \mu_2$.

93. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Assume that μ is non-atomic. Let $A \in \mathcal{A}$ and $0 \leq \alpha \leq \mu(A)$.

    Show that given $0 < \epsilon < \mu(A)$ there is $B \subset A$ such that $0 < \mu(B) < \epsilon$. Show that there is $B \subset A$, $B \in \mathcal{A}$ such that $\mu(B) = \alpha$. In particular the range of μ is the interval $[0, \mu(\Omega)]$. In fact, the range of μ on any set $A \in \mathcal{A}$ is $[0, \mu(A)]$.

94. Consider $\Omega = \{1, 2, \cdots\}$; $\mathcal{A} =$ power set of Ω; $\mu(n) = 1/2^n$ for $n \geq 1$. What is the range of μ?

95. Before any confusion sets in, recall that long ago we had the concept of atoms in a σ-field, smallest non-empty sets. More precisely, let $\mathcal{A}$ be a σ-field of subsets of Ω. A set $A \in \mathcal{A}$ is called an $\mathcal{A}$-atom if (i) $A \neq \emptyset$ and (ii) $B \subset A$, $B \in \mathcal{A}$ implies $B = \emptyset$ or $B = A$. Let now μ be a σ-finite measure on $\mathcal{A}$. It is not necessary that μ-atom is $\mathcal{A}$-atom.

    Let $\Omega = R$, $\mathcal{A}$ be the collection of countable snbsets of R and their complements. μ be thee measure hich assigns zero to countable sets and one to the others. Then R is an μ-atom but not $\mathcal{A}$-atom. Singleton sets are $\mathcal{A}$-atoms but not μ-atoms.

    Show that if $\mathcal{A}$ is countably generated and if A is a μ-atom, then there is $B \subset A$ which is an $\mathcal{A}$-atom and $B = A$ $[\mu]$.

96. As you know, the (abstract?) integration theory was developed based on Lebesgue's theory of integral on R, which itself was based on Borel's extension of concept of length on the line. Borel was motivated by number theoretic considerations. Here is Borel's theorem This is precursor to Kolmogorov's strong law of large numbers as well as Birkhoff's Ergodic theorem, both have profound importance.

Let μ be a probability on $(R, \mathcal{B})$, supported on a bounded interval with $\int x d\mu = 0$. Consider $\mu^\infty = \mu_1 \otimes \mu_2 \otimes \cdots$ where each $\mu_i = \mu$. Consider $(R^\infty, \mathcal{B}^\infty, \mu^\infty)$. Define the coordinate functions

$$f_i(x_1, x_2, \cdots,) = x_i, \quad \text{for } i \geq 1.$$

Define the averages by $g_n = \frac{f_1 + \cdots + f_n}{n}$.

Show $\int g_n^4 \, d\mu^\infty \leq C/n^2$ for some number C. Expand the integrand; use exercise 88 when applicable and when it is not applicable use bound on support of μ.

Deduce that for every $\epsilon > 0$, $\sum\limits_n P(\frac{|g_n|}{n} > \epsilon) < \infty$. Conclude that $g_n/n \to 0$ almost surely $[\mu^\infty]$.

Do not assume that $\int x d\mu = 0$, say $\int x d\mu = m$. Continue to assume that μ has bounded support. Show that now $g_n/n \to m$ almost surely $[\mu^\infty]$.

Take the special case when μ puts mass $1/2$ at one and $1/2$ at zero. Write down the conclusion. Do the same when μ puts mass $9/10$ at zero and $1/10$ at one.

Consider the unit interval, Borel σ-field and Lebesgue measure. Let $X_n(\omega)$ be the proportion of the digit one among the first n places in the binary expansion of ω (take nonterminating expansion). Use exercise 81 to show that $X_n \to 1/2$ almost surely.

Consider the unit interval, Borel σ-eld and the Lebesgue measure. Let $X_n(\omega)$ be the proportion of the digit 7 among the first n places in the decimal expansion of ω (take nonterminating). show that $X_n \to 1/10$ almost surely.

One can show more with a little bit of extra work. Suppose you fix any finite sequence s of decimal digits of length k. Let $\omega \in (0,1)$ with decimal expansion $\cdot\omega_1\omega_2\cdots$. Say that the pattern s occurs at place n in ω if $(\omega_n, \omega_{n+1}, \cdots, \omega_{n+k-1}) = s$. Let $X_n(\omega)$ be the proportion of occurrence of the pattern s in the first n decimal places of ω. One can show that $X_n \to 1/10^k$ almost surely.

The moral is that for almost every number in $(0,1)$, each pattern of decimal digits occurs with the corect proportion. Such numbers are called normal numbers. In fact for almost every number in $(0,1)$ the following is true: for every integer $r \geq 2$, each pattern of digits occurs with the correct (?) proportion. Such numnbers are called strongly normal, normal to every base. This is Borel's theorem.

97. You know that if you have a continuous function $f : R \to R$ such thhat $f(x+y) = f(x) + f(y)$ for all $x, y \in R$, then there is a umber a such that $f(x) = ax$ for all x. Would the conclusion be true without the assumption of continuity?

    No. Here is an example. Remember Hamel basis (exercise 84). Fix a Hamel basis B and an element $u \in B$. Define $f(x)$ to be the coefficient of u when x is expressed as a finite rational linear combination of elements of B.

    Alright, would something less than continuity do? Yes, if $f : R \to R$ is Borel and $f(x+y) = f(x) + f(y)$ holds for all x and y, then f is actually continuous and hence f is as described earlier. Here is outline of proof.

    First some topological notions are needed. A subset $A \subset R$ is said to be no-where dense if $\overline{A}$, closure of A, does not contain any non-empty open set. A set $A \subset R$ is of first category if it is a countable union of no-where dense sets. Clearly, a countable union of sets of first category is again of first category (This is not true for no-where dense sets, take singleton rationals). Translate of a first category set is again of first category (This is true for no-where dense sets too). There are two interesting facts due to Baire.

    First is Baire category theorem: R is not a first category set, that is, if $A \subset R$ is a first category set, then $R - A \neq \emptyset$. Second: Given any Borel set B, there is an open set U such that the symmetric difference $B \Delta U$ is a first category set. Prove it as follows: for open sets it is true; if true for a sequence of sets then true for their union; if true for a set then true for its complement.

    Returning to our problem, if we show a first category set A such that f restricted to $R - A$ is continuous, then we are done. Indeed, take $x_n \to x$, then the set $\cup_n (A + x_n) \cup (A + x)$ is of first category and hence pick a point a outside this set. Then all the points $(a - x_n)$ and $a - x$ are outside A and hence $f(a - x_n) \to f(a - x)$. Use the equation $f(a - x_n) + f(x_n) = f(a) = f(a - x) + f(x)$ to conclude $f(x_n) \to f(x)$.

    Enumerate open intervals with rational end points, say, $\{I_n : n \geq 1\}$, set $B_n = f^{-1}(I_n)$, get open U_n as above for B_n, set $A = \cup(B_n \Delta U_n)$. Observe $f^{-1}(I_n) \cap (R - A) = U_n \cap (R - A)$ and use this to deduce that f restricted to $(R - A)$ is continuous.

    Observe that the same argument shows that a Borel homomorphism between two complete separable metric groups is continuous.

98. Here is a little more about the tail sets, a verbal explanation and a localization.

Consider $(R^\infty, \mathcal{B}^\infty)$. Use the notation of exercise 89. Every set $A \in \mathcal{F}^n$ has the following property: $x \in A$, x and y differ at most in the first n coordinates, then $y \in A$. Do this for generating family and proceed.

Show that every tail set A has the following property, to be denoted by (*): If $x \in A$, x and y difffer in at most finitely many coordinates, then $y \in A$. This captures the essence of a tail set. More precisely, if $A \in \mathcal{B}^\infty$ and has the property (*), then A is indeed a tail set. To argue this, fix n. Let $\theta \in R^n$ be the point with all coordinates zero. Just as we showed that sections are measurable in discussing finite products, argue that the section $A_\theta = \{(y_1, y_2, \cdots) : (\theta, y_1, y_2, \cdots) \in A\} \in \mathcal{B}^\infty$. argue that our set A is nothing but $R^n \times A_\theta$ and is hence (?) in $\mathcal{F}^n$.

Suppose now you have a product measure $\mu = \mu_1 \otimes \mu_2 \otimes \cdots$. You saw in exercise 89 that μ is trivial on the tail σ-field. We can localize this. Suppose S is a Borel set in the real line and for each i, we have $\mu_i(S) = 1$. Then, clearly, $\mu(S^\infty) = 1$. Let us say that a set $A \subset S^\infty$ is S-tail if the following holds. Firstly, $A \in \mathcal{B}^\infty$. Secondly, $x, y \in S^\infty$; $x \in A$; x and y differ in finitely many coordinates; implies $y \in A$. Thus A is a tail set within S^∞. Since $\mu(S^\infty) = 1$, it is natural to expect that such an S-tail set must be trivial. Prove it. (Fix any special point $a \in S$, fix $n \geq 1$. Set $\theta \in R^n$ to be the point with all coordinates a, set A_θ to be the section of A at θ as done above, set $B_n = R^n \times A_\theta$ as above, set $B = \cup_n B_n$. Then B is a tail set in R^∞, zero-one law applies and $A = B \cap S^\infty$. In a sense B is 'tailification' of A.)

Since we did not do general product spaces, we had to fall back on R^∞.

99. You have seen some important consequences of product construction.

For example exercise 96, the normal number theorem of Borel, is purely a statement about numbers in the interval $(0, 1)$. We know some nonrational numbers which are normal (Champernowne's constant, Chaitin's constant, Erdos-Copeland constant etc) . We do not know if numbers like π and e, with which we are familiar, are normal.

Another example is exercise 89, Kolmogorov's zero-one law — which exhibits emergence of a dichotomy. Here is one more general principle that emerges, a consequence of the zero-one law.

For each $i \geq 1$, let μ_i be a probability on $(R, \mathcal{B})$ and $(R^\infty, \mathcal{B}^\infty, \mu)$ be the product space with product measure $\mu = \mu_1 \otimes \mu_2 \otimes \cdots$. Let us assume that each μ_i is concentrated on a countable set $D_i \subset R$. In other words, each μ_i is discrete as defined in exercise 78 (or atomic as defined in exercise 92).

Exercise 89 tells you that either $\sum x_i$ converges for $[\mu]$-almost every point $(x_1, x_2, \cdots) \in R^\infty$ or for almost no point. Let us assume that it converges for almost every point. Thus $f(x_1, x_2, \cdots) = \sum x_i$ is a legitimate measurable function on R^∞ (Define the value of f to be your favourite number at a point where the series does not converge). Let ν be the measure induced on $(R, \mathcal{B})$ by f. Recall, it is the measure defined as $\nu(B) = \mu(f^{-1}(B))$ for Borel $B \subset R$. (exercise 23).

Can we say anything about the measure ν? B. Jessen and A. Wintner tell us that ν is either discrete or continuous singular or absolutely continuous. By exercise 78, every measure on R is a sum of three measures; a discrete measure, a continuous singular measure and an absolutely continuous measure. What this result tells you is that our ν is 'pure'. Remember, there are no conditions on the component measures μ_i except that they are discrete.

Here is how you prove. Let $G \subset R$ be a countable group such that $\mu_i(G) = 1$ for all i. Show such a thing exists.

Let now $B \subset R$ be a Borel set such that $x \in B, g \in G$ implies $x + g \in B$. Using the previous exercise, deduce that $\nu(B) = 0$ or 1.

For any Borel set $B \subset R$, the set $B + G = \{x + y : x \in B, y \in G\}$ is a Borel set. Show this.(exercise 15).

If B is countable, so is $B + G$; if B is Lebesgue null, so is $B + G$.

For any Borel B, show that $\nu(B + G)$ is either one are zero.

There are exactly two possibilities: either for some countable set S, we have $\nu(S + G) = 1$ or for every countable set S we have $\nu(S + G) = 0$. In the first caser ν is discrete. In the second case it is continuous.

In the second case, either $\nu(S + G) = 1$ for some Lebesgue null set S or $\nu(S + G) = 0$ for every Lebesgue null set S. In the first subcase ν is continuous singular and in the second subcase, it is absolutely continuous.

100. Consider a locally compact metric group G with (left) Haar measure λ. Consider the complex Hilbert space $H = L^2(G, \lambda)$; where the inner product is $\langle \varphi, \psi \rangle = \int \varphi(x)\overline{\psi(x)} d\lambda(x)$ giving raise to usual L^2 norm $||\varphi||^2 = \langle \varphi, \varphi \rangle$. For each $g \in G$, let U_g be the linear operator on H defined by $U_g \varphi(x) = \varphi(gx)$. Show that this takes you from H onto H and preserves the inner product. Thus it is a unitary operator. Show that $g \mapsto U_g$ is a representation of the group G. This means $U_{gh} = U_g U_h$, $U_e = I$ (decipher the notation). It is strongly continuous, which means that if $g_n \to g$, then for every $\varphi \in H$ we have $U_{g_n}\varphi \to U_g \varphi$ in H. Show this only for continuous functions with compact support. Taking G to be the real line R, show this for all $\varphi \in L^2$.

101. The set of natural numbers N has a natural linear order ($\leq$). The countably infinite system $(N, \leq)$ has the property: for any $x \in N$, the set $\{y : y \leq x\}$ is finite.

Question: Is it possible to prescribe a linear order $\leq$ on $I = [0, 1]$ such that the uncountable system $(I, <)$ has the property: for any $x \in I$, the set $\{y : y \leq x\}$ is countable (finite or countably infinite). Unfortunately, the usual order is not helpful. Let us assume that there is such an order. Following the Polish mathematician S. M. Ulam, we shall show that there is no probability defined on all subsets of I giving zero mass to singletons. In particular, you can not define Lebesgue measure on all subsets of I.

If possible, let μ be a probability defined for all subsets of I, which gives zero mass to singletons. For each $x \in I$ let φ_x be a one-one function from the countable set $\{y : y \leq x\}$ into/onto $\{1, 2, \cdots\}$. For $y \in I$ and $n \geq 1$, define the set, $S(y, n) = \{x : \varphi_x(y) \text{ is defined and } = n\}$. This matrix of sets has uncountably many rows indexed by $y \in I$ but countably many columns indexed by $n \geq 1$.

Show that for fixed y, union of these countably many sets $\{S(y, n) : n \geq 1\}$ is co-countable. Indeed if x is such that $y \leq x$ then φ_x is defined at y and takes some value n and hence the union of these sets $\{S(y, n) : n \geq 1\}$ is the complement of the countable set $\{x : x < y\}$. Usnig that singletons — and hence countable sets — have measure zero deduce that $S(y, n)$ has positive measure for at least one n. Conclude that there is some n_0 such that the sets $S(y, n_0)$ have positive measure for uncountably many values of y. (If you mark those sets which have positive measure, then each row gets at least one mark and hence there are uncountably many marks. Since the number of columns are countable, one column must get uncountably many marks).

Since the functions φ_x are one-one, sets in a column are disjoint. That is, for each fixed n, the sets $\{S(y, n) : y \in I\}$ are disjoint. Indeed, if a point x is in two of these sets corresponding to y_1 and y_2 then $\varphi_x(y_1) = \varphi_x(y_2) = n$ contradicting φ_x is one-to-one map.

Show that there can not be uncountably many disjoint sets of positive measure and hence the conclusions of the earlier two paragraphs are contradictory.

What is the status of the hypothesis that lead to this conclusion? It can not be proved as a theorem in set theory, but it is consistent with the usual (?) axioms.

This then explains (at last!) the reasons for our not taking power-set as domain of definition of measures.