Construction of Haar measure (contd)

We had shown that $\mu^*$ is a left-invariant measure on $\mathcal{A}$. We will now show that $\mu^*$ gives a finite measure to every compact set and a positive measure to every nonempty open set.

Suppose $K \subseteq G$ is compact. $\mathcal{U}_0$ covers $K$, so some finitely many sets in $\mathcal{U}_0$, say $U_1, \ldots, U_l$ cover $K$. Since $\lambda$ of every set in $\mathcal{U}_0$ is finite,

$$\mu^*(K) \leq \sum_{i=1}^l \lambda(U_i) < \infty$$

Suppose $U \subseteq G$ is open, and $U \neq \emptyset$. Using local compactness, get nonempty open $H$ such that $H \subseteq \overline{H} \subseteq U$ and $\overline{H}$ is compact. For calculating $\mu^*(\overline{H})$, it suffices to consider only finite covers, since any countably infinite cover has a finite subcover, which gives only a better (lower) potential value for $\mu^*(\overline{H})$. Let $\epsilon > 0$ be given. Get $U_1, \ldots, U_l \in \mathcal{U}_0$ covering $\overline{H}$ such that $\sum_{i=1}^l \lambda(U_i) < \mu^*(\overline{H}) + \epsilon$.

Note that $\lambda$ is defined only on $\mathcal{U}$.

$$\lambda(H) \leq \lambda\left(\bigcup_{i=1}^l U_i\right) \quad \text{(monotonicity)}$$

$$\leq \sum_{i=1}^l \lambda(U_i) \quad \text{(finite subadditivity)}$$

$$\leq \mu^*(\overline{H}) + \epsilon$$

$$\leq \mu^*(U) + \epsilon \quad \text{(monotonicity)}$$

Since this holds for all $\epsilon > 0$, $\mu^*(U) \geq \lambda(H) > 0$.

$\mu^*$ is a left-invariant Haar measure on the Borel $\sigma$-field of $G$.

Uniqueness

In our current setup (a locally compact metrizable topological group $G$), we will show that if $\mu$ and $\nu$ are two left-invariant Haar measures, then $\mu$ can be scaled so that for all continuous functions with compact support $f : G \to \mathbb{C}$, we have $\int f \, d\mu = \int f \, d\nu$. A positive finite scalar multiple of a left-invariant Haar measure is also a left-invariant Haar measure.

The proof we give is a tricky proof due to Von Neumann, though more straightforward proofs exist.

Suppose $\mu, \nu$ are left-invariant Haar measures on $G$. Fix $\phi : [0, \infty)$, a continuous function with compact support which is not the identically zero function. This can be done using the metric and local compactness. $\phi$ is necessarily bounded. If $\phi$ takes the value $a$ somewhere, with $a > 0$, then $\{x : \phi(x) > a/2\}$ is nonempty open, so has positive measure. $\int \phi \, d\mu > 0$, and $\int \phi \, d\nu > 0$. We may scale $\mu$ and $\nu$ such that

$$\int \phi \, d\mu = \int \phi \, d\nu = 1$$

Define $\Delta : G \to [0, \infty)$,

$$\Delta(g) = \int \phi(xg^{-1}) \, d\mu(x)$$
Lemma. $\Delta$ is well-defined and is a continuous function of compact support.

Proof. To show that $\Delta$ is well-defined, we need to show that for each $g \in G$, the integral used to define $\Delta(g)$ is finite. Let $K \subseteq G$ be a compact set such that $\phi$ is zero outside $K$. $Kg$ is also compact, and so has finite $\mu$ measure (but $\mu(Kg)$ need not be equal to $\mu(K)$; $\mu$ is only left-invariant). To obtain $\Delta(g)$ we are integrating a bounded nonnegative function which is zero outside $Kg$, and so $\Delta(g)$ is finite.

To show continuity of $\Delta$, let $\{g_n\}_{n=1}^{\infty}$ be a sequence converging to $g$. We will show that

$$\hat{K} = Kg \cup \bigcup_{n \in \mathbb{N}} Kg_n$$

is compact. Take any sequence in $\hat{K}$. We need to find a convergent subsequence. If infinitely many terms occur in $Kg$ or in some $Kg_n$, then we are done. Otherwise, only finitely many terms are in $Kg$, and only finitely many terms are in $Kg_n$, for each $n$. Then we can find a subsequence whose $k$th term is in $Kg_{n_k}$, where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence. From this we can extract a further subsequence which converges to a point in $Kg$.

Now we can show that $\{\Delta(g_n)\}_{n=1}^{\infty}$ converges to $\Delta(g)$ using the usual DCT argument, using $M1_{\hat{K}}$ as the integrable bound, where $M$ is a bound on $\phi$. \qed

Lemma. For all $f : G \to \mathbb{C}$ continuous functions with compact support,

1. $\int f(x) \ d\mu(x) = \int f(x^{-1})\Delta(x^{-1}) \ d\mu(x)$.
2. $\int f(x) \ d\mu(x) = \int f(x^{-1})\Delta(x^{-1}) \ d\nu(x)$.
3. $\int f(x) \ d\mu(x) = \int f(x)\Delta(x)\Delta(x^{-1}) \ d\mu(x)$.
4. $\int f(x) \ d\mu(x) = \int f(x)\Delta(x)\Delta(x^{-1}) \ d\nu(x)$.
5. $\int f(x) \ d\mu(x) = \int f(x) \ d\nu(x)$.

Proof. We will prove 1 and 2 simultaneously. Fubini’s theorem will be used freely. Everything is bounded and has compact support, so is integrable. In what follows, $\rho$ can be taken to be either $\mu$ or $\nu$. $\rho = \mu$ proves 1 and $\rho = \nu$ proves 2.

$$\int f(x) \ d\mu(x)$$
$$= \left[ \int \phi(y) \ d\rho(y) \right] \left[ \int f(x) \ d\mu(x) \right]$$
$$= \int \left[ \int f(y^{-1}x) \ d\mu(x) \right] \phi(y) \ d\rho(y)$$
$$= \int \left[ \int f(y^{-1}x) \ d\mu(x) \right] \phi(y) \ d\rho(y)$$
$$= \int \left[ \int f(y^{-1}x)\phi(y) \ d\rho(y) \right] \ d\mu(x)$$
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$$= \int \left[ \int f(y^{-1})\phi(xy) \ d\rho(y) \right] \ d\mu(x)$$
$$= \int f(y^{-1}) \left[ \int \phi(xy) \ d\mu(x) \right] \ d\rho(y)$$
$$= \int f(y^{-1}) \Delta(y^{-1}) \ d\rho(y)$$

(replace $x$ by $y^{-1}x$, use left-invariance of $\mu$)

(replace $y$ by $xy$, use left-invariance of $\rho$)

We now prove 3 and 4, using 1 and 2. Again, $\rho$ can be taken to be either of $\mu$ or $\nu$. $\rho = \mu$ proves 3 and $\rho = \nu$ proves 4.

$$\int f(x)\Delta(x)\Delta(x^{-1}) \ d\rho(x)$$
\[
\int g(x^{-1}) \Delta(x^{-1}) \, d\rho(x) = \int g(x) \, d\rho(x) = \int f(x^{-1}) \Delta(x^{-1}) \, d\rho(x) = \int f(x) \, d\rho(x)
\]
(where \(g(x) = f(x^{-1}) \Delta(x^{-1})\))

(3) gives \(\Delta(x) \Delta(x^{-1}) = 1\) for all \(x \in G\). If not, suppose for some \(x\), \(\Delta(x) \Delta(x^{-1}) = a > 1\) \((a < 1\) is similar). Let \(U = \{x : \Delta(x) \Delta(x^{-1}) > \frac{1+a}{2}\}\). \(U\) is nonempty open. By choosing a suitable continuous function \(f\) with compact support contained in \(U\), we can get a contradiction to 3.

Since \(\Delta(x) \Delta(x^{-1}) = 1\) for all \(x \in G\), (4) gives 5.

Suppose we know that \(\mu\) and \(\nu\) are regular from below, that is,

\[\rho(A) = \sup\{\rho(K) : K \text{ compact}, K \subseteq A\}\]

for \(\rho \in \{\mu, \nu\}\). For example, this is true if \((G, d)\) is complete and separable. If \(K\) is a compact set, we can get a sequence of continuous functions of compact support decreasing to \(1_K\), each bounded between 0 and 1. DCT will give \(\mu(K) = \nu(K)\). Then regularity will give \(\mu = \nu\).