Fubini's theorem for σ-finite measures

Fubini's theorem for the nonnegative and integrable cases continues to hold for σ-finite measures:

Theorem (Fubini's theorem; nonnegative functions)**.** Suppose $(\Omega_1, \mathscr{A}_1, \mu_1)$ and $(\Omega_2, \mathscr{A}_2, \mu_2)$ are σ-finite measure spaces. Let

$$(\Omega, \mathscr{A}, \mu) = (\Omega_1, \mathscr{A}_1, \mu_1) \otimes (\Omega_2, \mathscr{A}_2, \mu_2)$$

be their product. Let $f : \Omega \to \bar{\mathbb{R}}$ be nonnegative measurable. Then

- For all $\omega_1 \in \Omega_1$, $f^{\omega_1} := \lambda\omega_2.f(\omega_1, \omega_2)$ is measurable on $(\Omega_2, \mathscr{A}_2)$.
- For all $\omega_2 \in \Omega_2$, $f_{\omega_2} := \lambda\omega_1.f(\omega_1, \omega_2)$ is measurable on $(\Omega_1, \mathscr{A}_1)$.
- $\lambda\omega_1.\left(\int f^{\omega_1}\ d\mu_2\right)$ is measurable on $(\Omega_1, \mathscr{A}_1)$.
- $\lambda\omega_2.\left(\int f_{\omega_2}\ d\mu_1\right)$ is measurable on $(\Omega_2, \mathscr{A}_2)$.

Finally,

$$\int f\ d\mu = \int \lambda\omega_1.\left(\int f^{\omega_1}\ d\mu_2\right)\ d\mu_1 = \int \lambda\omega_2.\left(\int f_{\omega_2}\ d\mu_1\right)\ d\mu_2$$

Theorem (Fubini's theorem; integrable functions)**.** Suppose $(\Omega_1, \mathscr{A}_1, \mu_1)$ and $(\Omega_2, \mathscr{A}_2, \mu_2)$ are σ-finite measure spaces. Let

$$(\Omega, \mathscr{A}, \mu) = (\Omega_1, \mathscr{A}_1, \mu_1) \otimes (\Omega_2, \mathscr{A}_2, \mu_2)$$

Suppose $f \in \mathcal{L}^1(\Omega, \mathscr{A}, \mu)$. Then

- For all $\omega_1 \in \Omega_1$, f^{ω_1} is measurable on $(\Omega_2, \mathscr{A}_2)$.
- For all $\omega_2 \in \Omega_2$, f_{ω_2} is measurable on $(\Omega_1, \mathscr{A}_1)$.
- For almost every $\omega_1 \in \Omega_1$ (w.r.t. μ_1), f^{ω_1} is μ_2-integrable. For ω_1 for which f^{ω_1} is not integrable, put $\int f^{\omega_1}\ d\mu_2 = 0$. $\lambda\omega_1.\left(\int f^{\omega_1}\ d\mu_2\right)$, defined with this convention, is measurable and μ_1-integrable.
- The corresponding mirror statement: For almost every $\omega_2 \in \Omega_2$ (w.r.t. μ_2), f_{ω_2} is μ_1-integrable. For ω_2 for which f_{ω_2} is not integrable, put $\int f_{\omega_2}\ d\mu_1 = 0$. $\lambda\omega_2.\left(\int f_{\omega_2}\ d\mu_1\right)$, defined with this convention, is measurable and μ_2-integrable.

Finally,

$$\int f\ d\mu = \int \lambda\omega_1.\left(\int f^{\omega_1}\ d\mu_2\right)\ d\mu_1 = \int \lambda\omega_2.\left(\int d_{\omega_2}\ d\mu_1\right)\ d\mu_2$$

The statements are the same as before, with "finite measure spaces" replaced by "σ-finite measure spaces". The proofs we gave earlier hold for the σ-finite case too; we never used the measure being finite. For example, we did not consider decreasing sequences of sets.

Integration is "area under the curve"

We now apply our knowledge of product measures and Fubini's theorem to say that integration corresponds to calculating the "area under the curve", the intuition we originally started with. The following theorem is about general measure spaces; the case of $(\mathbb{R}, \mathscr{B}, \lambda)$, where $\mathscr{B}$ is the Borel σ-field on $\mathbb{R}$ and λ is the Lebesgue measure, corresponds to our usual notion of area.

Theorem. Let $(\Omega, \mathscr{A}, \mu)$ is a σ-finite measure space. Let $f : \Omega \to [0, \infty)$ be measurable. Consider the product space $(\Omega, \mathscr{A}, \mu) \otimes (\mathbb{R}, \mathscr{B}, \lambda)$. Let

$$G = \{(x, y) \in \Omega \times \mathbb{R} : 0 \leq y \leq f(x)\}$$

Then $G \in \mathscr{A} \otimes \mathscr{B}$ and

$$\int f \, d\mu = (\lambda \times \mu)(G)$$

Here G is the region under the curve, and $(\mu \times \lambda)(G)$ is its area.

Proof. Let $\phi_1, \phi_2, \phi : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$\phi_1((x, y)) = f(x), \ \phi_2((x, y)) = -y, \ \phi = \phi_1 + \phi_2$$

For any interval $I \subseteq \mathbb{R}$, $\phi_1^{-1}(I) = f^{-1}(I) \times \mathbb{R} \in \mathscr{A} \otimes \mathscr{B}$. $\phi_2^{-1}(I) = \Omega \times (-I) \in \mathscr{A} \otimes \mathscr{B}$. So ϕ_1 and ϕ_2 are measurable. ϕ being their sum is also measurable.

$$G = \big\{(x, y) \in \Omega \times \mathbb{R} : 0 \leq y \leq f(x)\big\} = \big\{a \in \Omega \times \mathbb{R} : \phi(a) \geq 0\big\} \cap \big(\Omega \times [0, \infty)\big)$$

So $G \in \mathscr{A} \otimes \mathscr{B}$.

$$\begin{aligned}
(\mu \times \lambda)(G) &= \int 1_G \, d(\mu \times \lambda) \\
&= \int \lambda x. \left(\int (1_G)^x \, d\lambda\right) \, d\mu && \text{(Fubini's theorem for nonnegative functions)} \\
&= \int \lambda x. \left(\int 1_{G^x} \, d\lambda\right) \, d\mu \\
&= \int \lambda x. \left(\int 1_{\{y:(x,y)\in G\}} \, d\lambda\right) \, d\mu \\
&= \int \lambda x. \left(\int 1_{[0,f(x)]} \, d\lambda\right) \, d\mu && \text{(definition of } G\text{)} \\
&= \int \lambda x.(f(x)) \, d\mu \\
&= \int f d\mu
\end{aligned}$$

□

Product of finitely many measure spaces

Just like we took the product of two measure spaces, we can take the product of any finite number of measure spaces. Here the complete details involved are a bit boring and conceptually there is nothing new, so we will not look at this in detail.

Theorem. Let $(\Omega_i, \mathscr{A}_i, \mu_i)$, $1 \leq i \leq n$ be σ-finite measure spaces. Let

- $\Omega = \Omega_1 \times \ldots \times \Omega_n$.

- $\mathcal{R} = \{A_1 \times \ldots \times A_n : \text{ each } A_i \in \mathscr{A}_i\}$.
- $\mathscr{A} = \sigma(\mathcal{R})$.

There is a unique measure $\mu : \mathscr{A} \to [0, \infty]_{\bar{\mathbb{R}}}$, such that for $A_1 \times \ldots \times A_n \in \mathcal{R}$,

$$\mu(A_1 \times \ldots \times A_n) = \prod_{i=1}^{n} \mu_i(A_i)$$

This μ is σ-finite.

Fubini's theorem holds for products of n measure spaces. This will be stated in terms of a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, by saying that the integral of a function with respect to the product measure is the same as repeated integration performed in the order $\pi(1), \ldots, \pi(n)$, for any permutation π.

Theorem. Suppose $(\Omega, \mathscr{A}, \mu)$ is a measure space, and $T : \Omega \to \Omega$ is an isomorphism of measure spaces (that is, T is bijective, for all $A \in \mathscr{A}$, $T(A)$ and $T^{-1}(A)$ belong to $\mathscr{A}$, and $\mu(A) = \mu(T(A))$), then for all measurable f, $f \circ T$ is measurable. For f nonnegative or integrable, $\int f \circ T = \int f$.

Proof. For any interval $I \subseteq \mathbb{R}$, $(f \circ T)^{-1}(I) = T^{-1}(f^{-1}(I))$. Since $f^{-1}(I) \in \mathscr{A}$, $T^{-1}(f^{-1}(I)) \in \mathscr{A}$. If f is measurable, so is $f \circ T$.

For $A \in \mathscr{A}$, $\mu(T^{-1}(A)) = \mu(T(T^{-1}(A))) = \mu(A) = \mu(T(A))$. The proof is as usual: $1_A \circ T = 1_{T^{-1}(A)}$, and so $\int 1_A = \int 1_A \circ T$. By linearity, for any nonnegative simple function s, $\int s = \int s \circ T$. For any nonnegative measurable function f, by taking a sequence of simple functions increasing to it and using the monotone convergence theorem, $\int f = \int f \circ T$. For any integrable f, by looking at f^+ and f^-, we have that $f \circ T$ is integrable and $\int f = \int f \circ T$. □

This can be used to deal with product spaces: for example, if all the n measure spaces are the same, then the "transposition" map $\Omega \to \Omega$ which interchanges the ith component with the jth component is an example of such a T.

Towards Radon-Nikodym theorem

We have many ways of obtaining new measure spaces from old: restricting the basic set, restricting the σ-field, adding measures on the same set and σ-field, taking products, etc. We will give another way now. First, a matter of notation: for a measure space $(\Omega, \mathscr{A}, \mu)$, $f : \Omega \to \bar{\mathbb{R}}$, and $A \in \mathscr{A}$, by $\int_A f \, d\mu$ we mean $\int f 1_A \, d\mu$.

Theorem. Suppose $(\Omega, \mathscr{A}, \mu)$ is a σ-finite measure space, and $f : \Omega \to [0, \infty)$ measurable. Define $\nu : \mathscr{A} \to [0, \infty]_{\bar{\mathbb{R}}}$,

$$\nu(A) = \int_A f \, d\mu$$

Then ν is a σ-finite measure on $(\Omega, \mathscr{A})$.

Proof. Clearly $\nu(\emptyset) = 0$. Suppose $\mathcal{C} \subseteq \mathscr{A}$ is a countable collection of disjoint sets.

$$\begin{aligned}
\nu\left(\bigcup \mathcal{C}\right) &= \int f 1_{\bigcup \mathcal{C}} \, d\mu \\
&= \int f \sum_{C \in \mathcal{C}} 1_C \, d\mu \\
&= \int \sum_{C \in \mathcal{C}} f 1_C \, d\mu \\
&= \sum_{C \in \mathcal{C}} \int f 1_C \, d\mu \qquad \text{(use partial sums, linearity of integral, MCT)} \\
&= \sum_{C \in \mathcal{C}} \nu(C)
\end{aligned}$$

So ν is a measure.

Let $\mathcal{B} \subseteq \mathscr{A}$ be a countable collection of sets with finite μ measure whose union is Ω. Consider the countable collection of sets

$$\left\{B \cap \{x : f(x) \in [-n, n]\} : B \in \mathcal{B}, n \in \mathbb{N}\right\}$$

$|f 1_{B \cap \{x : f(x) \in [-n,n]\}}| \leq n 1_B$, so $\nu(\{B \cap \{x : f(x) \in [-n, n]\}) \leq n\mu(B)$. This countable collection of sets each with finite ν measure clearly covers Ω; so ν is σ-finite. □

Also, note that the above ν is finite if and only if f is integrable.

ν also has the property that for all $A \in \mathscr{A}$, if $\mu(A) = 0$, then $\nu(A) = 0$ (because if $\mu(A) = 0$, then $f 1_A$ is the zero function almost everywhere (w.r.t. μ), and so has μ integral 0). The above construction is the only way such measures can arise!

Theorem (Radon-Nikodym)**.** Suppose Ω is a nonempty set and $\mathscr{A}$ a σ-field on it. Suppose μ and ν are σ-finite measures on $(\Omega, \mathscr{A})$ such that for all $A \in \mathscr{A}$, $\mu(A) = 0$ implies $\nu(A) = 0$. Then

- There exists $f : \Omega \to \mathbb{R}$ nonnegative measurable such that for all $A \in \mathscr{A}$, $\nu(A) = \int_A f \, d\mu$.
- Such an f is unique upto a.e. equality (w.r.t. μ).
- f is integrable w.r.t. μ if and only if ν is a finite measure.

We will prove the theorem in the next class.