Measurable functions and simple functions

The class of all real measurable functions on $(\Omega, \mathscr{A})$ is too vast to study directly. We identify ways to study them via simpler functions or collections of functions. Recall that $\mathcal{L}$ is the set of all measurable functions from $(\Omega, \mathscr{A})$ to $\mathbb{R}$ and $\mathcal{E} \subseteq \mathcal{L}$ are all the simple functions.

Theorem. Suppose $f \in \mathcal{L}$ is bounded. Then there exists an increasing sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{E}$ whose *uniform* limit is f.

Proof. First assume that f takes values in $[0, 1)$. We divide $[0, 1)$ into 2^n intervals and use this to construct f_n:

$$f_n = \sum_{k=0}^{2^n-1} \frac{k}{2^n} 1_{f^{-1}\left([\frac{k}{2^n}, \frac{k+1}{2^n})\right)}$$

Whenever f takes a value in $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$, f_n takes the value $\frac{k}{2^n}$. We have

- For all n, $f_n \leq f$.
- For all n, x, $|f_n(x) - f(x)| \leq \frac{1}{2^n}$. This is clear from the construction.
- $f_n \in \mathcal{E}$, since f_n is a finite linear combination of indicator functions of sets in $\mathscr{A}$.
- $f_n \leq f_{n+1}$: For any x, if $f_n(x) = \frac{k}{2^n}$, then $f_{n+1}(x) \in \left\{\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right\}$.

So $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{E}$ converging uniformly to f.
Now for a general f, if the image of f lies in $[a, b)$, then let

$$g = \frac{f - a1_\Omega}{b - a}$$

(note that $a1_\Omega$ is the constant function a)
Im $g \subseteq [0, 1)$, so by the above we have $\{g_n\}_{n=1}^{\infty}$ from $\mathcal{E}$ increasing uniformly to g. Let

$$f_n = a1_\Omega + (b - a)g_n$$

Then $\{f_n\}_{n=1}^{\infty}$ is a sequence from $\mathcal{E}$ increasing uniformly to f. □

Theorem. Suppose $f \in \mathcal{L}$ is nonnegative (that is, $f(x) \geq 0$ for all $x \in \Omega$). Then there exists an increasing sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{E}$ whose pointwise limit is f.

Proof. As in the previous proof, we divide the range of f into intervals of length $\frac{1}{2^n}$. However, f could be unbounded and we may need infinitely many of such intervals, which will not help in defining a simple function. So at the nth stage, we divide only $[0, n)$, and not the entire $[0, \infty)$. More precisely, let

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} 1_{f^{-1}\left([\frac{k}{2^n}, \frac{k+1}{2^n})\right)}$$

- For all n, $f_n \leq f$. This is clear from the construction of f_n (note that $f \geq 0$).
- For all n, x, if $f(x) < n$, then $|f_n(x) - f(x)| < \frac{1}{2^n}$.

- $f_n \in \mathcal{E}$, since f_n is a finite linear combination of indicator functions of sets in $\mathscr{A}$.
- $f_n \leq f_{n+1}$: if $f_n(x) = 0$ then clearly $f_n(x) \leq f_{n+1}(x)$, otherwise $f(x) < n < n+1$ and from the construction it is easy to see that $f_n(x) \leq f_{n+1}(x)$.

Now for any $x \in \Omega$, there is some integer m such that $m > f(x)$. For all $n \geq m$, $|f_n(x) - f(x)| < \frac{1}{2^n}$, and so $\lim_{n\to\infty} f_n(x) = f(x)$. □

For $f : \Omega \to \mathbb{R}$, define $f^+, f^- : \Omega \to \mathbb{R}$

$$f^+(x) = \max\{f(x), 0\}$$
$$f^-(x) = -\min\{f(x), 0\}$$

Both f^+ and f^- are nonnegative, and $f = f^+ - f^-$. If f is measurable, so are both f^+ and f^-. Every measurable function is the difference of two nonnegative measurable functions. This is useful because by the above theorem, nonnegative measurable functions are easier to deal with than general measurable functions.

Theorem. Suppose Ω is a nonempty set and $\mathscr{A}$ is a σ-field on it. $\mathscr{F}$ is a field such that $\sigma(\mathscr{F}) = \mathscr{A}$. Suppose $\underline{\underline{F}}$ is a collection of real measurable functions on Ω which is a vector space under the natural operations and closed under monotone pointwise limits. Suppose for all $A \in \mathscr{F}$, $1_A \in \underline{\underline{F}}$. Then $\underline{\underline{F}} = \mathcal{L}$, the set of all real measurable functions.

Proof. We first show that indicator functions of all sets in $\mathscr{A}$ belong to $\underline{\underline{F}}$. Let

$$\mathscr{B} = \{A \in \mathscr{A} : 1_A \in \underline{\underline{F}}\}$$

We are given that $\mathscr{F} \subseteq \mathscr{B}$. Since $\underline{\underline{F}}$ is closed under monotone pointwise limits, $\mathscr{B}$ is a monotone class. By the monotone class theorem, $\mathscr{B}$ is a σ-field, and so $\mathscr{B} = \mathscr{A}$. So $\underline{\underline{F}}$ has:

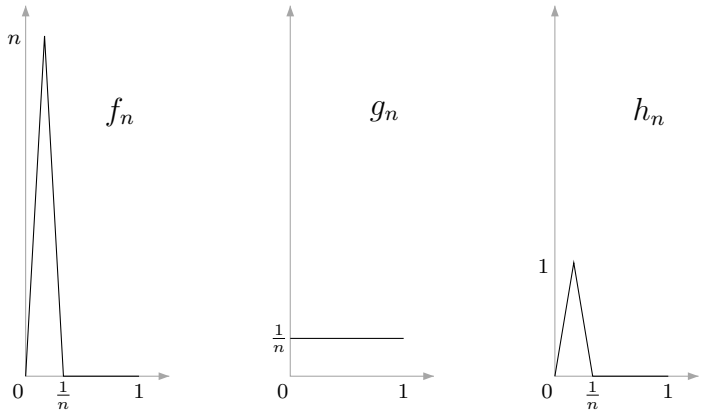
- 1_A for all $A \in \mathscr{A}$.
- By taking finite linear combinations of the above (since $\underline{\underline{F}}$ is a vector space), all simple functions.
- By taking increasing pointwise limits of simple functions, all nonnegative measurable functions.
- By taking differences of nonnegative measurable functions, all measurable functions. □

Sometimes we are concerned only about bounded measurable functions. We have a similar theorem for that case. We need a definition first.

Definition. A sequence of $\Omega \to \mathbb{R}$ functions $\{f_n\}_{n=1}^{\infty}$ is said to converge *bounded pointwise* (abbreviated as bp) to $f : \Omega \to \mathbb{R}$ if

- $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise.
- There is a $C \in \mathbb{R}$ such that for all n and x, $|f_n(x)| \leq C$.

In other words, bounded pointwise convergence is just pointwise convergence along with uniform boundedness of the sequence of functions. Here are some examples of $[0,1] \to \mathbb{R}$ functions which illustrate various modes of convergence.



$\{f_n\}_{n=1}^{\infty}$ converges to the zero function pointwise, but not bounded pointwise or uniformly. $\{g_n\}_{n=1}^{\infty}$ converges to the zero function uniformly and bounded pointwise. $\{h_n\}_{n=1}^{\infty}$ converges to the zero function bounded pointwise but not uniformly.

Theorem. Suppose Ω is a nonempty set and $\mathscr{A}$ is a σ-field on it. $\mathscr{F}$ is a field such that $\sigma(\mathscr{F}) = \mathscr{A}$. Suppose $\underline{\underline{F}}$ is a collection of *bounded* real measurable functions on Ω which is a vector space under the natural operations and closed under *bounded* pointwise limits. Suppose for all $A \in \mathscr{F}$, $1_A \in \underline{\underline{F}}$. Then $\underline{\underline{F}}$ is the set of all bounded real measurable functions.

Proof. As before, we first show that indicator functions of all sets in $\mathscr{A}$ belong to $\underline{\underline{F}}$. Let

$$\mathscr{B} = \{A \in \mathscr{A} : 1_A \in \underline{\underline{F}}\}$$

We are given that $\mathscr{F} \subseteq \mathscr{B}$. Since $\underline{\underline{F}}$ is closed under bounded pointwise limits, $\mathscr{B}$ is a monotone class (all indicator functions are bounded by 1). By the monotone class theorem, $\mathscr{B}$ is a σ-field, and so $\mathscr{B} = \mathscr{A}$. So $\underline{\underline{F}}$ has:

- 1_A for all $A \in \mathscr{A}$.
- By taking finite linear combinations of the above (since $\underline{\underline{F}}$ is a vector space), all simple functions.
- By taking increasing bounded pointwise limits of simple functions, all nonnegative bounded measurable functions (if $|f(x)| \leq C$ for all x, we can get an increasing sequence of simple functions converging to it pointwise (even uniformly, in fact), all of which take values between 0 and C).
- By taking differences of nonnegative bounded measurable functions, all bounded measurable functions (if f is bounded, so are f^+ and f^-). $\square$

Defining integration

We have studied the properties of the class of measurable functions, now we define their integrals. Suppose $(\Omega, \mathscr{A}, \mu)$ is a measure space, and $f : \Omega \to \mathbb{R}$ is measurable. How can we define $\int_\Omega f d\mu$? If we could define the integral for f^+ and f^-, we could define

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

because we want integration to be a linear operation. So we have reduced the problem from general measurable functions to nonnegative measurable functions. If f is nonnegative, how can we define $\int_\Omega f d\mu$? If s is a nonnegative simple function, say taking values $a_1, \ldots, a_k$, then we can define

$$\int_\Omega s d\mu = \sum_{i=1}^{k} a_i \mu\big(f^{-1}(\{a_i\})\big)$$

(this corresponds to our notion of "area under the curve")

Since we want integration to be monotonic, we should have

$$\int_\Omega f d\mu \geq \sup \left\{ \int_\Omega s d\mu : s \text{ simple }, s \leq f \right\}$$

Since f is nonnegative, we know that nonnegative simple functions s with $s \leq f$ exist, in fact we can get a sequence of them converging to f. Though it is not immediately clear, it makes sense to define the integral of f by taking the above equation to be an equality. We will see this in the next class.