More on σ-fields

Recall our field of finite disjoint unions of intervals of the type $(a, b]$, denoted by $\mathscr{F}$. Let $\mathscr{B} = \sigma(\mathscr{F})$ be the σ-field generated by $\mathscr{F}$. This known as the Borel σ-field on $\mathbb{R}$, and sets in it are called Borel sets. Any subset of $\mathbb{R}$ one can think of is likely to belong to $\mathscr{B}$; it is quite hard to come up with a set not in $\mathscr{B}$, though such sets exist.

One can also define $\mathscr{B}$ as the smallest σ-field containing all intervals in $\mathbb{R}$. For example, for $a, b \in \mathbb{R}$, $(a, b) \in \mathscr{B}$ because

$$(a, b) = \bigcup_{n \in \mathbb{N}} \left(a, b - \tfrac{1}{n}\right]$$

Similarly it can be shown that all intervals belong to $\mathscr{B}$, and so $\mathscr{B}$ is also the σ-field generated by all intervals.

Here are some examples of Borel sets:

- Any singleton, $\{a\}$. $\{a\} = \bigcap_{n \in \mathbb{N}} \left(a - \frac{1}{n}, a\right]$
- Any countable set, say $\mathbb{Q}$. A countable set is a countable union of singletons.
- The set of transcendental numbers. Its complement, the set of algebraic numbers, is countable.

Definition. Let Ω be a set, and $\mathscr{A}$ a σ-field on it. A *measure* on $(\Omega, \mathscr{A})$ is a function

$$\mu : \mathscr{A} \to [0, \infty]_{\bar{\mathbb{R}}}$$

such that

- $\mu(\emptyset) = 0$
- If $\mathcal{C} \subseteq \mathscr{A}$ is a countable collection of disjoint sets, then

$$\mu\left(\bigcup \mathcal{C}\right) = \sum_{C \in \mathcal{C}} \mu(C)$$

  This property is called countable additivity.

Note that unlike before we do not have to put a separate condition for $\bigcup \mathcal{C} \in \mathscr{A}$, this follows from the definition of a σ-field.

Definition. μ is called a *probability measure* if $\mu(\Omega) = 1$.

Definition. μ is called a *finite measure* if $\mu(\Omega) < \infty$.

Definition. μ is called a *σ-finite measure* if there exists a countable $\mathcal{C} \subseteq \mathscr{A}$ such that $\bigcup \mathcal{C} = \Omega$ and $\mu(C) < \infty$ for all $C \in \mathcal{C}$. In other words, μ is σ-finite if Ω can be written as a union of countably many sets of finite measure.

In the above definition, the sets in $\mathcal{C}$ can be made disjoint if needed: enumerate them as $C_1, C_2, \ldots$, and define

$$D_i = C_i \setminus \left[\bigcup_{1 \le j < i} C_j\right]$$

D_i and D_j are disjoint for $i \neq j$, and $\bigcup_{i \in \mathbb{N}} D_i = \Omega$.

$$\mu(C_i) = \mu(D_i) + \mu(C_i \setminus D_i)$$

So $\mu(D_i) \le \mu(C_i) < \infty$.

Extending the measure from the field to the σ-field

We will prove a major result which allows us to extend a measure from a field to a σ-field uniquely.

Theorem (Carathéodory extension theorem)**.** Let Ω be a set and $\mathscr{F}$ a field on it. Let μ be a measure on $\mathscr{F}$. Assume μ is σ-finite, that is, Ω can be expressed as countable union of sets in $\mathscr{F}$ with finite measure[1]. Then there is a unique measure $\tilde{\mu}$ on $\sigma(\mathscr{F})$ such that μ and $\tilde{\mu}$ agree on $\mathscr{F}$. Clearly $\tilde{\mu}$ is σ-finite.

(Proof done later.)

Corollary. Let $F : \mathbb{R} \to \mathbb{R}$ be non-decreasing and right-continuous. Define F at ∞ and $-\infty$ as before by taking limits. There is a unique measure μ on the Borel σ-field of $\mathbb{R}$ such that for any $a < b$ in $\bar{\mathbb{R}}$,

$$\mu((a, b]) = F(b) - F(a)$$

Definition. If we take F to be the identity function in the above, the measure so obtained is called the *Lebesgue measure*.

Lemma. Let Ω be a set, $\mathscr{A}$ a field on it, and μ a measure on $(\Omega, \mathscr{A})$. In what follows below, all sets are from $\mathscr{A}$:

1. If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
2. If $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets increasing to A, then $\mu(A_n) \uparrow \mu(A)$ as $n \to \infty$.
3. If $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets decreasing to A and $\mu(A_1) < \infty$, then $\mu(A_n) \downarrow \mu(A)$ as $n \to \infty$.

(Note that we have to require that $A \in \mathscr{A}$. If $\mathscr{A}$ is a σ-field, then this follows from each A_n being in $\mathscr{A}$.)

Proof.

1. $\mu(B) = \mu(A) + \mu(B \setminus A)$. So $\mu(B) \geq \mu(A)$.
2. By the above, $\{\mu(A_n)\}_{n=1}^{\infty}$ is an increasing sequence. Let $A_0 = \emptyset$. For $n \in \mathbb{N}$, let $B_n = A_n \setminus A_{n-1}$. Then A is the disjoint union of $\{B_n\}_{n=1}^{\infty}$, and so
$$\mu(A) = \sum_{i \in \mathbb{N}} \mu(B_i)$$
$\bigcup_{i=1}^{k} B_i = A_i$, so $\sum_{i=1}^{k} \mu(B_i) = \mu(A_k)$. Taking limits, $\lim_{k\to\infty} \mu(A_k) = \mu(A)$.
3. $\{\mu(A_n)\}_{n=1}^{\infty}$ is a decreasing sequence. $A_1 \setminus A$ is a disjoint union of the collection $\{A_i \setminus A_{i+1} : i \in \mathbb{N}\}$. So
$$\mu(A_1 \setminus A) = \sum_{i \in \mathbb{N}} \mu(A_i \setminus A_{i+1})$$
Since $\mu(A_1 \setminus A) < \infty$, the tail sums of the above converge to 0.
$$\sum_{i=n}^{\infty} \mu(A_i \setminus A_{i+1}) = A_n \setminus A$$

[1] we need to state the definition here, since the above definition applies only for measures on σ-fields

So $\lim_{n\to\infty} \mu(A_n \setminus A) = 0$. Adding $\mu(A)$ to both sides, $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

□

The third part of this lemma needs $\mu(A_n)$ to be finite for some n. For example, consider $(\mathbb{R}, \mathscr{F}, \mu)$, where $\mu((a,b]) = b - a$. $\mu((n,\infty)) = \infty$ for all n. These sets decrease to the empty set, but $\mu(\emptyset) = 0$.

We now proceed towards the proof of the Carathéodory extension theorem by showing that proving it for finite measures suffices.

Lemma. Suppose we have proven the Carathéodory extension theorem for finite measures. Then the theorem for general measures follows.

Proof. Let Ω, $\mathscr{F}$, μ be as in the statement of the theorem, with μ σ-finite. Take a collection of sets of $\mathscr{F}$, $\{\Omega_i\}_{i\in\mathbb{N}}$ such that $\bigcup_{i\in\mathbb{N}} \Omega_i = \Omega$ and each $\mu(\Omega_i) < \infty$. As observed before, we may assume that $\{\Omega_i\}_{i\in\mathbb{N}}$ is a collection of disjoint sets (the argument also works for a field). For each $i \in \mathbb{N}$, define $\mu_i : \mathscr{F} \to [0,\infty]_{\bar{\mathbb{R}}}$

$$\mu_i(A) = \mu(A \cap \Omega_i)$$

Note that $A \cap \Omega_i \in \mathscr{F}$. It is easy to check that μ_i is a measure, using μ being a measure.

Moreover, each μ_i is a finite measure, since $\mu_i(\Omega) = \mu(\Omega_i) < \infty$. Applying the finite case of the theorem to all these measures, we get measures on the σ-field

$$\tilde{\mu}_i : \sigma(\mathscr{F}) \to [0,\infty]_{\bar{\mathbb{R}}}$$

extending μ_i.

Define $\tilde{\mu} : \sigma(\mathscr{F}) \to [0,\infty]_{\bar{\mathbb{R}}}$ by

$$\tilde{\mu}(A) = \sum_{i\in\mathbb{N}} \tilde{\mu}_i(A)$$

Clearly $\tilde{\mu}(\emptyset) = 0$. We now show that $\tilde{\mu}$ is countably additive. This follows from countable additivity of $\tilde{\mu}_i$ and being able to interchange the order of summation for nonnegative quantities. Let $\mathcal{B} \subseteq \sigma(\mathscr{F})$ be a countable collection of disjoint sets.

$$\begin{aligned}
\tilde{\mu}\left(\bigcup \mathcal{B}\right) &= \sum_{i\in\mathbb{N}} \tilde{\mu}_i\left(\bigcup \mathcal{B}\right) && \text{(definition of } \tilde{\mu}) \\
&= \sum_{i\in\mathbb{N}} \sum_{B\in\mathcal{B}} \tilde{\mu}_i(B) && \text{(by countable additivity of } \tilde{\mu}_i) \\
&= \sum_{B\in\mathcal{B}} \sum_{i\in\mathbb{N}} \tilde{\mu}_i(B) && \text{(can interchange for nonnegative numbers)} \\
&= \sum_{B\in\mathcal{B}} \tilde{\mu}(B) && \text{(definition of } \tilde{\mu})
\end{aligned}$$

We now show that $\tilde{\mu}$ extends μ. Let $A \in \mathscr{F}$.

$$\begin{aligned}
\tilde{\mu}(A) &= \sum_{i\in\mathbb{N}} \tilde{\mu}_i(A) && \text{(definition of } \tilde{\mu}\text{)} \\
&= \sum_{i\in\mathbb{N}} \mu_i(A) && (\tilde{\mu}_i \text{ extends } \tilde{\mu}) \\
&= \sum_{i\in\mathbb{N}} \mu(A \cap \Omega_i) && \text{(definition of } \mu_i\text{)} \\
&= \mu(A) && \text{(countable additivity of } \mu \text{ on } \mathscr{F}\text{)}
\end{aligned}$$

We have shown that μ has an extention to $\sigma(\mathscr{F})$. We now have to show that this extension is unique. Let $\nu : \sigma(\mathscr{F}) \to [0,\infty]_{\bar{\mathbb{R}}}$ be any measure extending μ. For each $i \in \mathbb{N}$, define $\nu_i : \sigma(\mathscr{F}) \to [0,\infty]_{\bar{\mathbb{R}}}$ by

$$\nu_i(A) = \nu(A \cap \Omega_i)$$

ν_i is a measure on $(\Omega, \sigma(\mathscr{F}))$. For $A \in \mathscr{F}$,

$$\nu_i(A) = \nu(A \cap \Omega_i) = \mu(A \cap \Omega_i) = \mu_i(A)$$

Note that $A \cap \Omega_i \in \mathscr{F}$. ν_i and $\tilde{\mu}_i$ are both extensions of μ_i to $\sigma(\mathscr{F})$. Using the the uniqueness in the finite-measure part of the theorem, we get that $\nu_i = \tilde{\mu}_i$.

For any $A \in \sigma(\mathscr{F})$,

$$\nu(A) = \sum_{i\in\mathbb{N}} \nu(A \cap \Omega_i) = \sum_{i\in\mathbb{N}} \nu_i(A) = \sum_{i\in\mathbb{N}} \tilde{\mu}_i(A) = \tilde{\mu}(A)$$

So $\nu = \tilde{\mu}$. This proves uniqueness. $\square$