Multicut

Generalizes Multiway Cut we saw earlier

Given a graph G, k pairs $\{s_i, t_i\}$ of vertices and edge costs c_e. Find min cost $F \subseteq E(G)$ such that in $G - F$ s_i is disconnected from t_i $\forall i \in [k]$

LP:

$$\min \sum_e c_e x_e$$

$$\forall i \in [k] \quad \forall P \in \mathcal{P}_i \quad \sum_{e \in P} x_e \geq 1$$

$$x_e \geq 0 \quad \forall e$$

$\mathcal{P}_i$: set of all paths from s_i to t_i

Exponentially many constraints ⇒ Separation Oracle to solve the LP

- Given a feasible LP soln x $\quad V^* = \sum_e c_e x_e$

  $x_e \rightarrow$ length of e

  $c_e \rightarrow$ width of e

  $\therefore \; c_e x_e \rightarrow$ Volume of e

- $d : V \times V \rightarrow \mathbb{R}$ is the shortest path metric using above lengths

  $d(u,v)$ = length of the shortest u-v path.

- For $S \subseteq V(G)$ $\quad \delta(S) = \{ e \mid$ exactly one endpt of $e \in S \}$

- For s_i and $r \in [0,1]$ $\quad B_x(s_i, r) = \{ v \mid d(s_i, v) \leq r \}$

$$V_x(s_i, r) = \frac{V^*}{k} + \sum_{e \in B \times B} c_e x_e + \sum_{e \in \delta(B)} c_e (r - d_u)$$

$d_u = d(s_i, u)$

explained later

$r - d_u$: the portion of length x_e contained in B

Lemma: Given a feasible soln x to the LP, for vertex s_i one can find $r < 1/2$ such that

$$c(\delta(B(s_i, r))) \leq 2 \ln(k+1) \, V_x(s_i, r)$$

Theorem: Multicut has an $O(\log k)$ approximation

Proof:

- Given LP soln x
- $F \leftarrow \phi$
- while s_i, t_i are connected
  - Find r as per Lemma for s_i
  - $F \leftarrow F \cup \delta(B(s_i, r))$
  - Delete $B(s_i, r)$ from G
- return F

$B_i = B(s_i, \pi)$ $\quad F_i = \delta(B_i) \quad F = \cup F_i$

- F is a multicut because F_i cuts $s_i - t_i$
- Let V_i = total volume of all edges incident on B_i
  - Note: Any edge contributes to at most one V_i since B_i is deleted
  - $V_i \geq V(s_i, \pi) - V^*/k$ → extra term in defⁿ of $V(s_i, \pi)$
    ↑ volume of edges in F_i is partially included
  - By Lemma: $c(F_i) \leq 2\ln(k+1)\, V(s_i, \pi)$
    $\leq 2\ln(k+1)\left(V_i + V^*/k\right)$
  - $\sum_{i=1}^{k} V_i \leq V^* = \sum_e c_e x_e \leq OPT$
  - $c(F) \leq \sum 2\ln(k+1)\left(V_i + V^*/k\right) \leq 4\ln(k+1)\, V^* \leq 4\ln(k+1)\, OPT$

Lemma: Given a feasible soln x to the LP, for any vertex s_i we can find $r < 1/2$ such that

$$c(r) = c(\delta(B(s_i, r))) \leq 2 \ln(k+1)\, V_x(s_i, r) = V(r)$$

- We argue, $\underset{r \sim [0, 1/2)}{E}\left[c(r)/V(r)\right] \leq 2\ln(k+1)$

$\Rightarrow \exists\, r \in [0, 1/2)$ s.t. $c(r)/V(r) \leq 2\ln(k+1)$

We then give an algorithm to find r in polynomial time

- Let $f(x) = \ln(V(x))$ Assume it is differentiable (NOT TRUE)

- Mean Value Thm: $\exists\, x \in (0, 1/2)$ $f'(x) \leq f(1/2) - f(0) / 1/2 - 0$

$$V(x) = V^*/k + \sum_{e \in B \times B} c_e u_e + \sum_{e \in \delta(B)} c_e (x - d_u)$$

$\sum_e c_e u_e$

$$V(1/2) \leq V^*/k + V^*$$

$$V(0) = V^*/k$$

$$\therefore f(1/2) - f(0) = \ln(V(1/2)) - \ln(V(0))$$

$$= \ln\left(V(1/2)/V(0)\right)$$

$$\leq \ln\left(\frac{V^*(k+1)}{k} \cdot \frac{k}{V^*}\right)$$

$$= \ln(k+1)$$

$$-f'(x) = \frac{d}{dx}(\ln(V(x)))$$

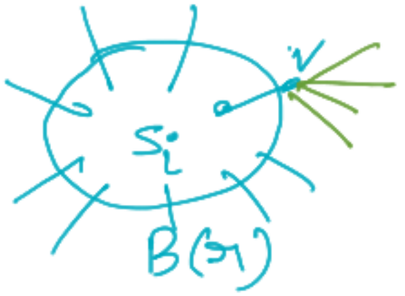
$$= \frac{d/dx\,(V(x))}{V(x)}$$

$$d/dx\,(V(x)) = c(\delta(B(s_i, x))$$

$$\| \quad c(x)$$

$$= c(x)/V(x)$$

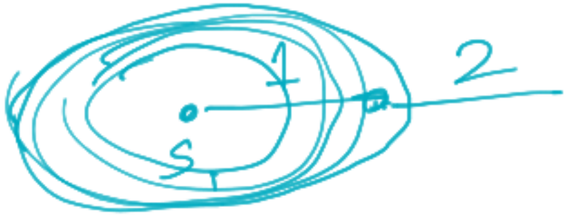
- But $V(r)$ is not differentiable, or even continuous
  - Discontinuity whenever $B(r)$ includes one new vertex
    - But at most n such vals of r
  - $V(r)$ is always increasing.
  - Sort vertices in $B(s_i, 1/2)$ by increasing dist from s_i



$s_i = v_0, v_1, \ldots, v_{\ell-1}$

$0 = r_0 \le r_1 \le \cdots \le r_{\ell-1} \le r_\ell = 1/2$

$\bar{r_j} = r_j - \varepsilon$ for some tiny $\varepsilon > 0$



$$\mathop{\mathbb{E}}_{r \sim (0,1/2)} \left[c(r)/V(r) \right] = \sum_{j=0}^{\ell-1} \int_{r_j}^{\bar{r}_{j+1}} \underbrace{\frac{c(r)}{V(r)}}_{\text{RV}} \underbrace{\frac{1}{1/2 - 0}}_{\text{PDF}} dr$$

Expected Value

$$\frac{1}{1/2} \sum_{j=0}^{l-1} \int_{x_j}^{x_{j+1}} \frac{c(x)}{V(x)}\, dx$$

↳ $V(x)$ or $\ln(V(x))$ is differentiable in $[x_j, x_{j+1}]$

$$= 2 \sum_{j=0}^{l-1} \left[\ln(V(x))\right]_{x_j}^{x_{j+1}}$$

$$\leq 2 \sum_{j=0}^{l-1} \left(\ln(V_{x_{j+1}}) - \ln(V_{x_j})\right)$$

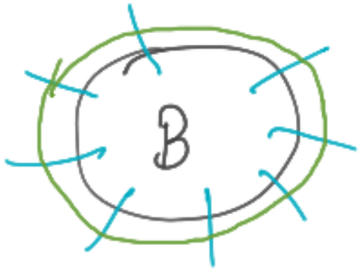
$$= 2\left(\ln(V(1/2)) - \ln(V(0))\right)$$

$$\leq 2\ln(k+1)$$

Hence $\exists\, x$ st $c(x)/V(x) \leq 2\ln(k+1)$

How do we compute this value of σ?

- if $\sigma \in [\sigma_j, \sigma_{j+1}]$
  - $c(\sigma)$ is constant in this interval
    so $S(B)$ is unchanged
  - $V(\sigma)$ is non-decreasing



$\therefore \quad c(\bar{\sigma}_{j+1}) / V(\bar{\sigma}_{j+1}) \leq 2 \ln(k+1)$

So enough to check $\{\bar{\sigma}_1, \bar{\sigma}_2, \ldots \bar{\sigma}_l\}$