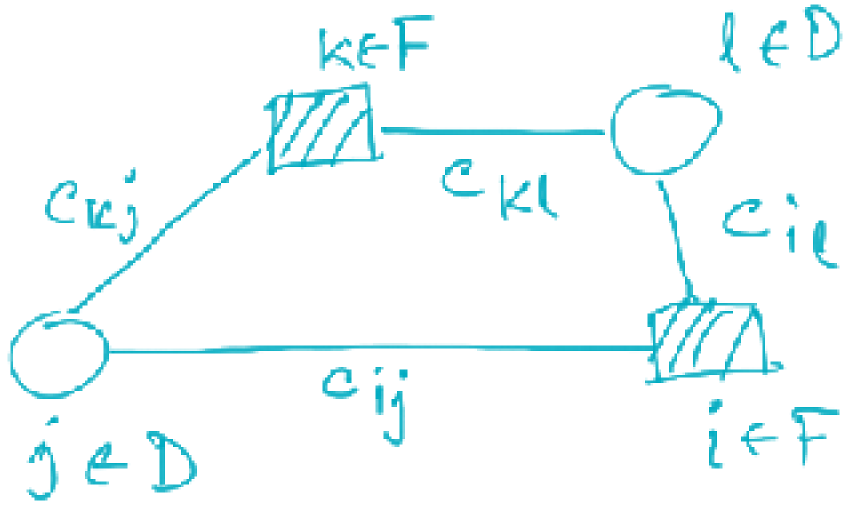
# Primal Dual Algorithms for Uncapacitated Facility Location

- We have $D \rightarrow$ clients or demands
  $F \rightarrow$ potential facilities

  Think of them as points in some metric space

- $i \in F$ has Opening Cost $f_i$
- if $j \in D$ is assigned $i \in F$ (must be open) then Travel cost $c_{ij}$

  Triangle Inequality




$c_{ij} \leq c_{kj} + c_{kl} + c_{il}$

- $\min \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in D} c_{ij} x_{ij}$

  $y_i$: 1/0: open $i \in F$

  $x_{ij}$: 1/0: assign $j$ to $i$

- $\forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \longrightarrow$ assign each client to some facility
- $\forall j \in D \ \forall i \in F \quad x_{ij} \leq y_i \longrightarrow$ assign to Open facility only

- $\min \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in D} c_{ij} x_{ij}$

Primal

- $\forall j \in D \quad \sum_{i \in F} x_{ij} \geq 1 \quad \} \; v_j$
- $\forall j \in D \;\; \forall i \in F \quad y_i - x_{ij} \geq 0 \quad \} \; w_{ij}$

$x_{ij} \leq y_i$

$$\max \sum_{j \in D} v_j \cdot 1 + \sum_{i,j} w_{ij} \cdot 0$$

Dual

$$\forall i \in F \quad \sum_j w_{ij} \leq f_i$$

$$\forall j \in D \quad v_j - w_{ij} \leq c_{ij}$$

$$v_j, w_{ij} \geq 0$$

# Primal Dual Algo:

- $S = D$, $T = \phi$
- $v = 0$, $w = 0$
- while some $j \in S$ is not neighbor of $T$ — i and j are neighbors if $v_j \geq c_{ij}$
  - increase uniformly $v_j \ \forall j \in S$, $w_{ij} \ \forall j \in S$ st $i \in N(j)$ (i may or may not be in T)
  - If $v_j \geq c_{ij}$ for some $i \in T$ $\quad S \leftarrow S - \{j\}$
  - If $\sum_j w_{ij} = f_i$ $\quad T \leftarrow T \cup \{i\}$
    $S \leftarrow S - N(i)$
- Prune T:
  - pick $i \in T$ and drop $i' \in T$ if $\exists\, j \in D$ st both $w_{ij}, w_{i'j} > 0$ } until $\forall j \in D$ there is exactly one $i$ st $w_{ij} > 0$

  This seems dangerous but it will be fine.
- Assign each $j \in D$ to the closest facility in T

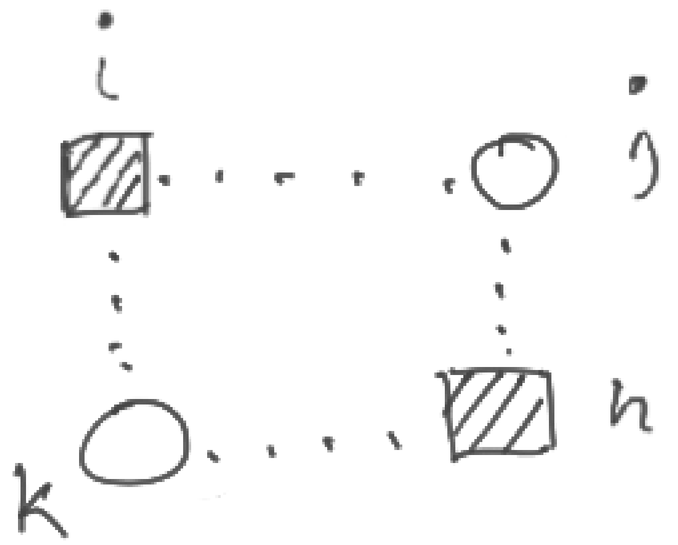
- Let $T' \leftarrow T$ after pruning
- <u>Lemma:</u> If $j \in D$ has no neighbor in $T'$ (i.e. $v_j < c_{ij} \; \forall i \in T'$)

  then $\exists\, i \in T'$ st $3v_j \geq c_{ij}$

<u>Proof:</u> - we stopped increasing $v_j$ because $v_j \geq c_{hj}$ for some $h \in F$

- $h \notin T'$ by assumption

  $\Rightarrow \exists\, i \in T'$ and $k \in D$ st $v_k > c_{ik}$, $v_k > c_{hk}$




- $v_j \geq v_k$ because:
  - we increase both uniformly
  - when we stop incrementing $v_j$ $h \in T$ and <u>we also stop inc $v_k$</u>

    if not, then $v_k < c_{ik}$ and $v_k < c_{hk}$

    $\Rightarrow$ we increment it only until $v_k = c_{ik}$ or $v_k = c_{hk}$

    $\Rightarrow$ contradiction to $v_k > c_{ik}$, $v_k > c_{hk}$

- $\Delta$ ineq: $c_{ij} \leq c_{ik} + c_{hk} + c_{hj} \leq 3v_j$

Thm: We obtain a 3-approximation

Pf: $\forall j \in D$ st $\exists i \in T'$ st $v_j \geq c_{ij}$

cost of opening $T'$ is accounted here

$$\sum_{i \in T'} f_i + \sum_{\substack{i \in T' \\ j \in A(i)}} c_{ij}$$

($A(i)$ : neighbors of $i$ in $D$)

$$= \sum_{i \in T'} \sum_{j \in A(i)} (w_{ij} + c_{ij})$$

(as $f_i = \sum w_{ij}$)

$$= \sum_{i \in T'} \sum_{j \in A(i)} v_j$$

(we set $w_{ij} = v_j - c_{ij}$)

$$= \sum_{j \in N(i) \text{ for some } i \in T'} v_j$$

(each $j \in D$ has at most one neighbor in $T'$ because of pruning)

And $\forall j \in D$ with no neighbor in $T'$, $\sum_{i \in T'} c_{ij} \leq 3 \sum v_j$

$\therefore$ total cost $\leq 3 \sum_{j \in D} v_j \leq 3\,OPT$