# Lecture 5: Training Deep Neural Networks II

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Chennai Mathematical Institute

## Advanced Machine Learning 2022

(based on slides by Madhavan Mukund)

# Ill conditioning

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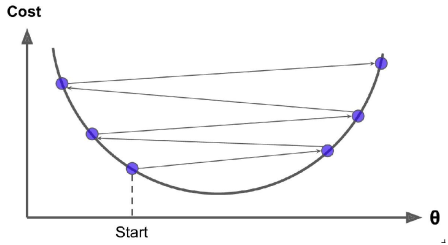

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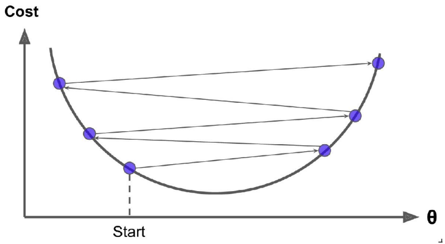

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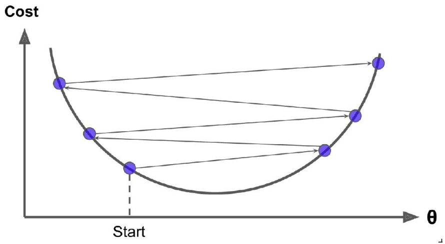

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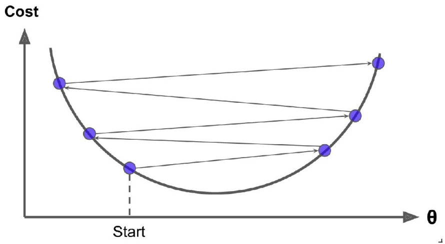

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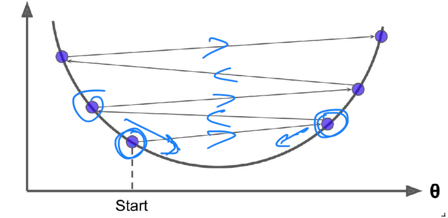

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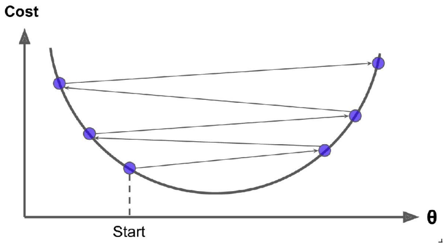

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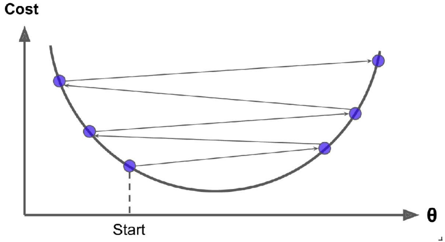
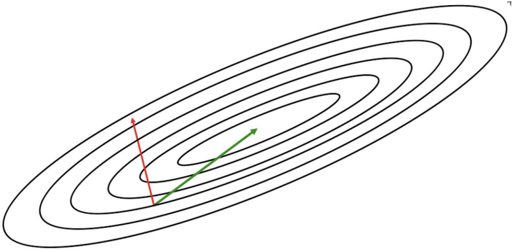
- Analyze $H_\theta$ to check for ill conditioning



# Directing gradient descent

- Locally steepest direction of descent may be far from the optimum
  - Elliptical contours vs circular contours



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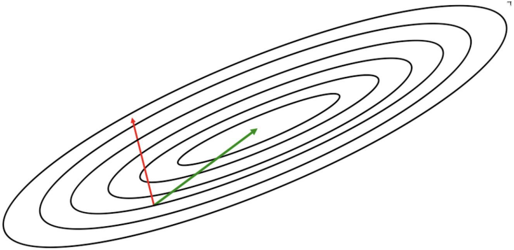

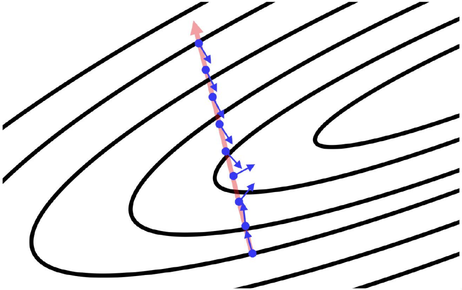

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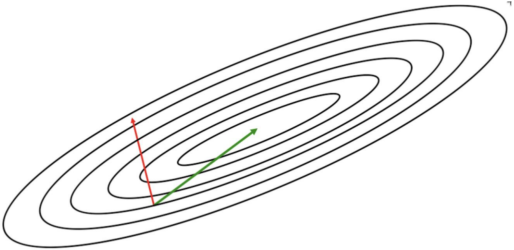

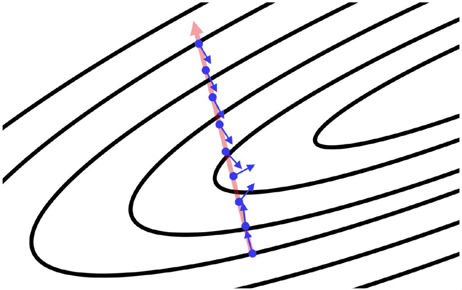

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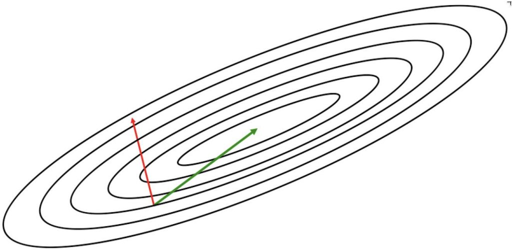
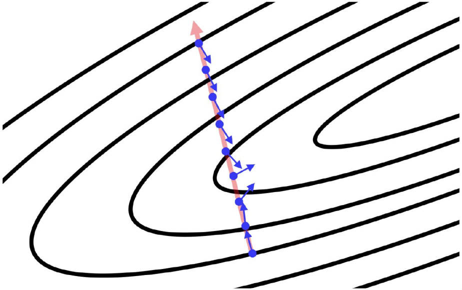
  - "Second order" methods are not used in practice
- Instead, heuristics like momentum and adaptive learning rates





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  - If gradient remains steady, velocity increases
- Update rule
  - $v \leftarrow \alpha v - \epsilon \nabla_\theta \left( \frac{1}{m} \sum_{i=1}^{m} L(f(x_i;\theta), y_i) \right)$
  - $\theta \leftarrow \theta + v$

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- Hyperparameter $\alpha \in [0, 1)$ — "friction", exponentially decaying history
  - With constant gradient $g$, in the limit $\frac{\epsilon g}{1-\alpha}$, geometric progression

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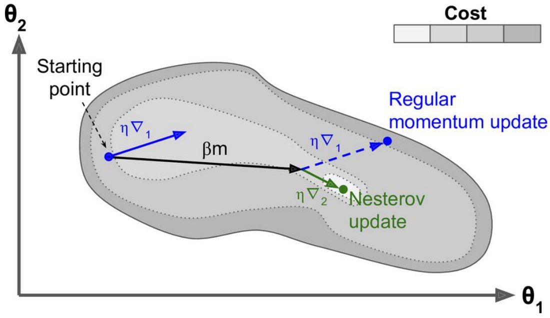
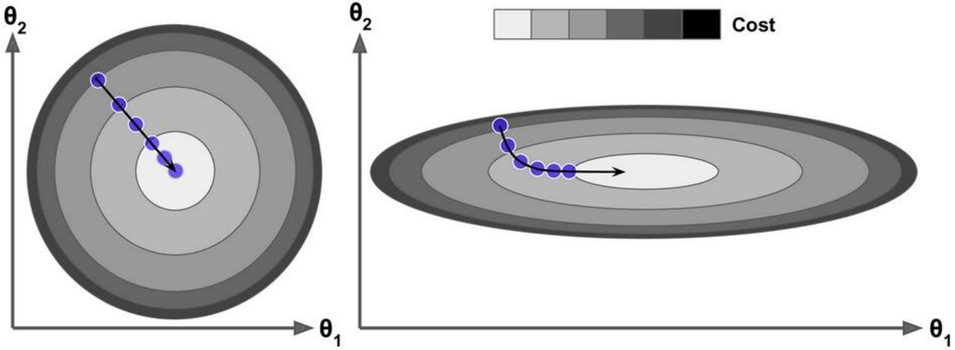
    $$\epsilon \nabla_\theta \left( \frac{1}{m} \sum_{i=1}^{m} L(f(x_i; \theta + \beta m), y_i) \right)$$
  - $\theta \leftarrow \theta + v$
- Controls sideways oscillations better



# Adjusting the trajectory

- If features have different scales, gradient descent is steeper in some dimensions
- How can we correct for this?



# Adagrad

- Adagrad update rule
  - $g \leftarrow \frac{1}{m} \nabla_\theta \sum_{i=1}^{m} L(f(x_i;\theta), y_i)$
  - $r \leftarrow r + g \cdot g$
  - $\Delta\theta \leftarrow \frac{\epsilon}{\delta + \sqrt{r}} \cdot g$, where $\delta \approx 10^{-7}$, for numerical stability
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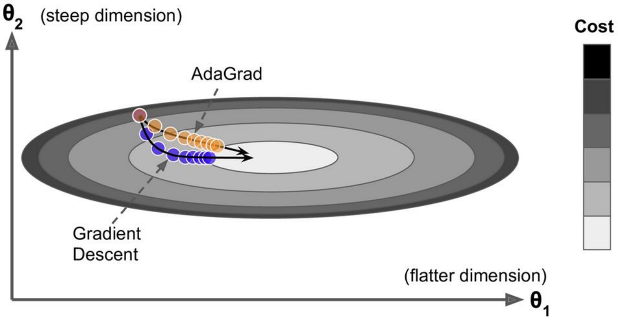

# Adaptive learning rates

## RMSProp

- Using entire history shrinks learning rate too much
- Exponentially decaying average, discard extreme past
- Update rule
  - $g \leftarrow \frac{1}{m} \nabla_\theta \sum_{i=1}^{m} L(f(x_i;\theta), y_i)$
  - $r \leftarrow \rho r + (1-\rho)(g \cdot g)$, where $\rho$ is decay rate
  - $\Delta\theta \leftarrow \frac{\epsilon}{\sqrt{\delta + r}} \cdot g$, where $\delta \approx 10^{-6}$
  - $\theta \leftarrow \theta + \Delta\theta$
- New hyperparameter $\rho$

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- Update rule
  - Step size $\epsilon$; two decay rates $\rho_1$, $\rho_2$; two moments, $s = r = 0$; time step $t = 0$
  - $g \leftarrow \frac{1}{m} \nabla_\theta \sum_{i=1}^{m} L(f(x_i; \theta), y_i)$
  - $s \leftarrow \rho_1 s + (1 - \rho_1) g$; $r \leftarrow \rho_2 r + (1 - \rho_2)(g \cdot g)$
  - Correct bias in first and second moments: $\hat{s} \leftarrow \frac{s}{1 - \rho_1^t}$, $\hat{r} \leftarrow \frac{r}{1 - \rho_2^t}$
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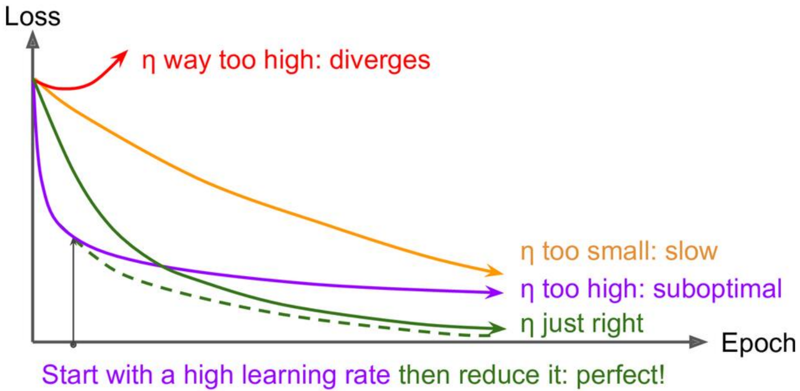
  - $\Delta\theta = -\epsilon \frac{\hat{s}}{\sqrt{\hat{r}} + \delta}$; $\theta \leftarrow \theta + \Delta\theta$
- No clear theoretical justification for combinining momentum and scaling
- Fairly robust with respect to values of hyperparameters

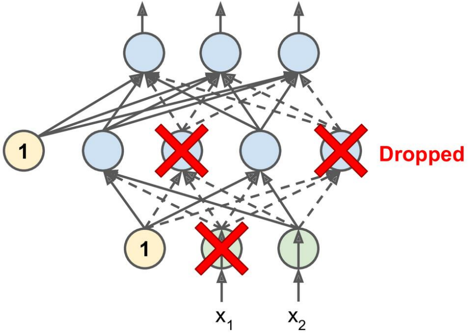
# Adaptive learning rates

- Choosing a fixed learning rate is hard
- Make a learning rate a function of iteration number
- Power scheduling, exponential scheduling, piecewise constant scheduling



# Regularization

- $\ell_1$ and $\ell_2$ regularization, as usual
- Dropout
  - Disable nodes with probability $p$
  - Analogy — multifunctional employees
  - Scale weights after training



# Local minima

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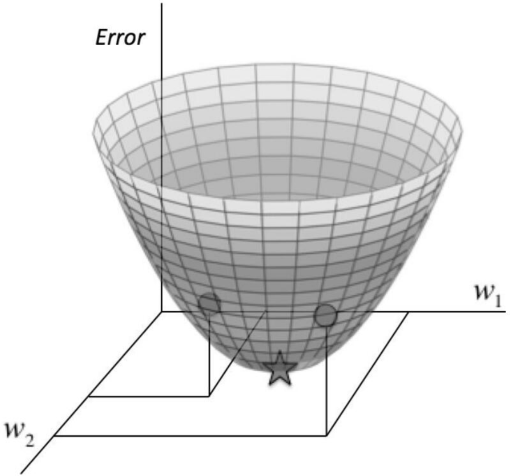
- Gradient descent converges to global optimum



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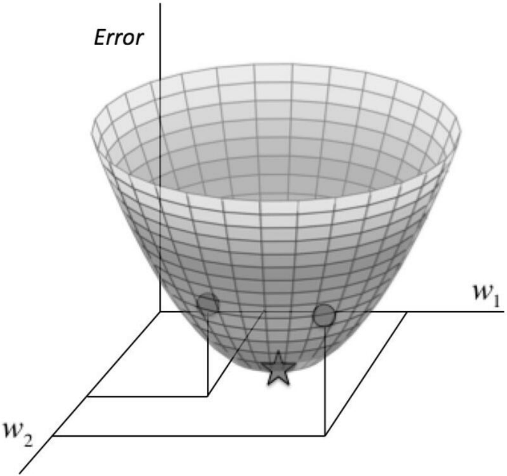

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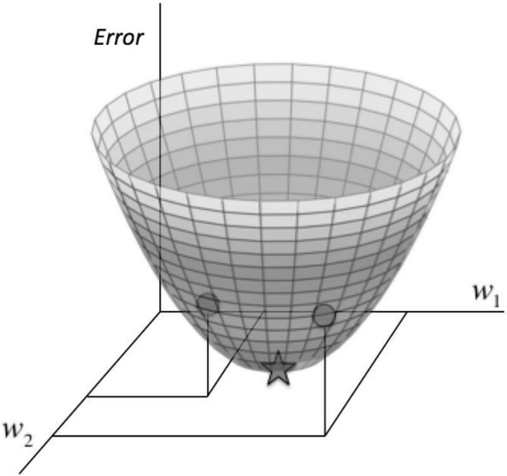

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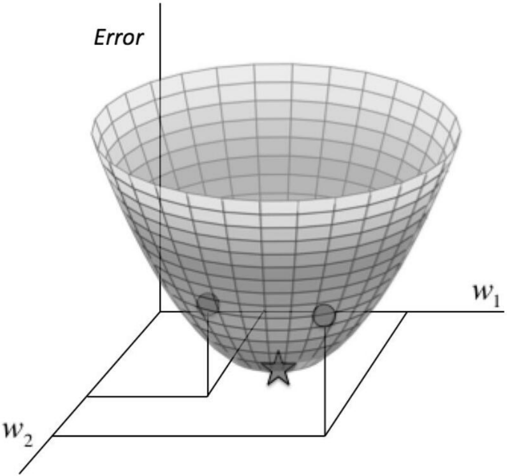
- How does it affect gradient descent?



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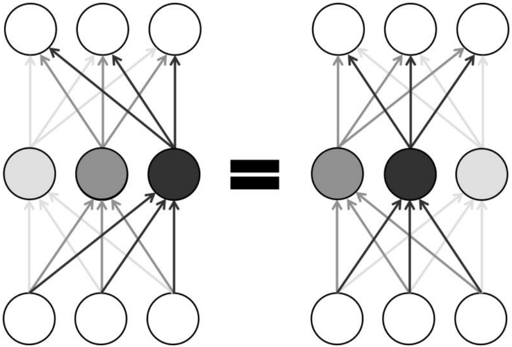

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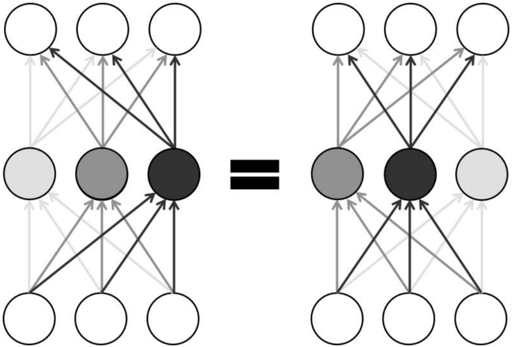

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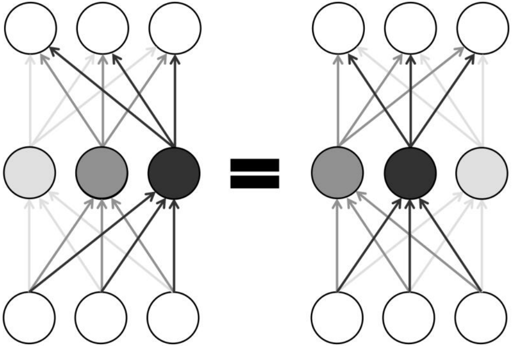

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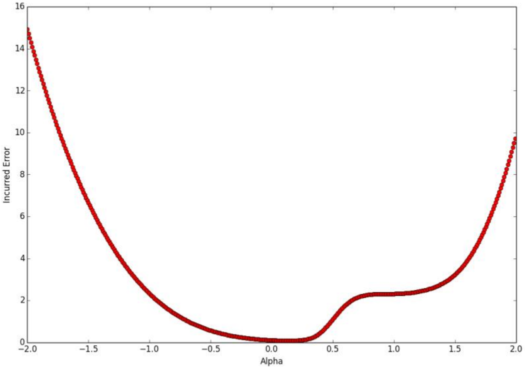

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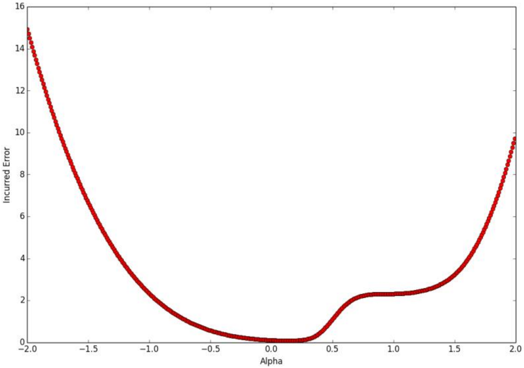

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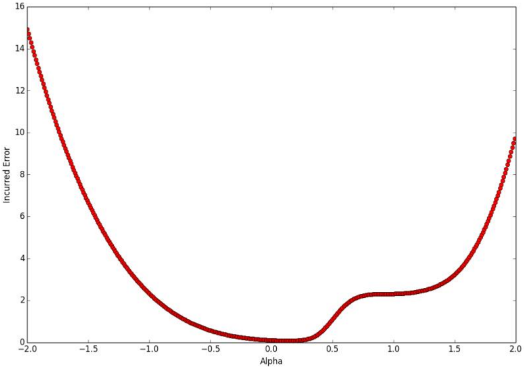
  - Are there problematic local minima along the path?
- Typically, no!



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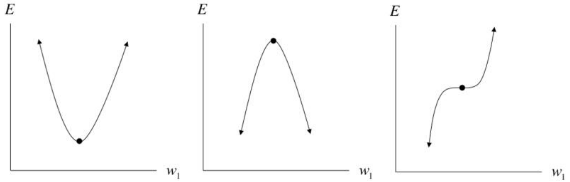

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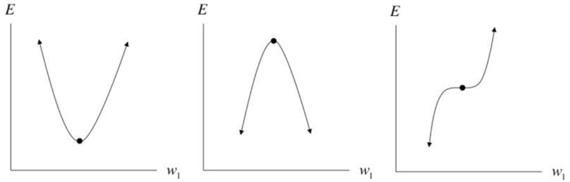

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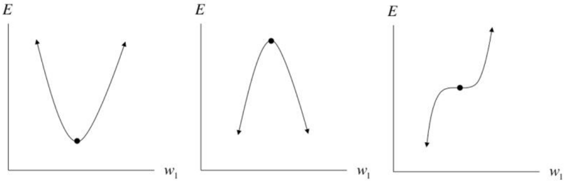


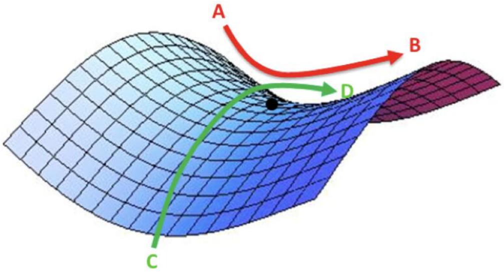

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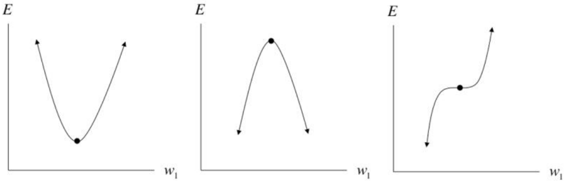


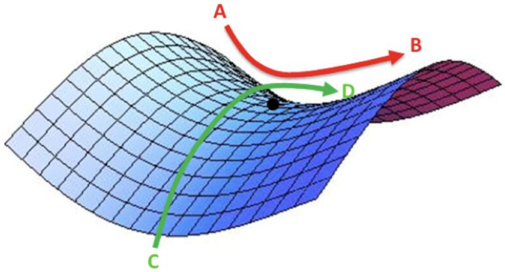

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- Solving directly for zero gradient is problematic