TORSORS
MIDTERM

Due on Monday October 22. More precisely, I expect you to hand in at least two problems per class starting Monday October 8 until Monday October 22. You may hand in any two problems you wish to (provided you haven't handed them in earlier). You are free to use results of other problems (whether you've done them or haven't) provided the quoted result is one that occurs earlier in the list. The last condition is to prevent circular arguments. I remind you that we will be having Friday classes on Friday afternoons for the next four weeks so I expect you to hand in two problems in class even on the Fridays that occur in the period Oct 8—Oct 22. You may, if you wish, hand in more solutions, but that will not lessen the requirement for the succeeding class. It will only mean you are done earlier.

Definitions and Preliminaries Recall that a map of schemes $f\colon X \to Y$ is fpqc if it is faithfully flat (or simply flat) and every quasi-compact open subset of Y is the image of a quasi- compact open subset of X. Equivalently, the map f is fpqc if it is *flat* and satisfies any of the equivalent conditions in Problem (6).

Given a map $f\colon X \to Y$ of topological spaces, the *higher direct images* $\mathbf{R}^i f_*\mathscr{F}$ of a quasi-coherent sheaf $\mathscr{F}$ on X are the right derived functors of f_*. In practical terms, these are computed by first finding an injective or flasque resolution $\mathscr{F} \to \mathscr{I}^\bullet$ and then computing the cohomology of the complex $f_*\mathscr{I}^\bullet$. Thus $\mathbf{R}^i f_*\mathscr{F} = H^i(f_*\mathscr{I}^\bullet)$. In the event $f\colon X \to Y$ is a map of noetherian the injectives can be taken to be injectives in the category of $\mathscr{O}_X$-modules, since these are flasque in the noetherian case. We will make the simplifying assumption that schemes that occur in the exercise are noetherian. Removing the noetherian hypothesis is possible, but requires work that is a distraction from the main theme of the course. Note that since f_* is left exact, $\mathbf{R}^0 f_* = f_*$.

If we have a cartesian square in the category of schemes $\mathbb{S}\mathrm{ch}$

(†)
$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

then for every quasi-coherent sheaf $\mathscr{F}$ on X and every integer i there is a natural map

$$u^*\mathbf{R}^i f_*\mathscr{F} \to \mathbf{R}^i {f'}_* v^*\mathscr{F}$$

called the *base change map of higher direct images.* If u is *flat*, the above map is an isomorphism for every quasi-coherent sheaf $\mathscr{F}$ and every i. In other words "higher direct images commute with flat base change". See Hartshorne, Proposition 9.3, page 255 for further details.

Let **Sets** denote the category of sets. Recall that singleton sets of the form $\{a\}$ are the terminal objects (final objects) in **Sets**. We fix once and for all such a final

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object and denote it $\bigstar$. Thus given a set A, there is exactly one map $A \to \bigstar$. To work around annoying trivialities, we allow this even when $A = \emptyset$.

On an *affine* scheme $X = \operatorname{Spec} A$, the global sections functor $\Gamma(X, -)$ on quasi-coherent sheaf is exact. On the category of A modules the *sheafification functor* $M \mapsto \widetilde{M}$ is also exact. The two processes are inverses of each other in an obvious sense. Thus if $\mathscr{C}^\bullet$ is a complex of quasi-coherent shaves of X and $C^\bullet$ the complex of A-modules given by $C^i := \Gamma(X, \mathscr{C}^i)$, then

$$\Gamma(X, H^i(\mathscr{C}^\bullet)) = \mathrm{H}^i(C^\bullet)$$

and

$$H^i(\mathscr{C}^\bullet) = \mathrm{H}^i(C^\bullet)^\sim.$$

Finally, recall that if $f\colon X \to Y$ is a map of schemes, and $Z \hookrightarrow Y$ is a closed subscheme with quasi-coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_Y$, then the closed subschema $f^{-1}(Z)$ of X is defined to be the closed subscheme whose quasi-coherent ideal sheaf is $\mathscr{I}\mathscr{O}_X$. It is well known (and easily checked) that in this case the topological subspace (of the topological space of X) which underlies the scheme $f^{-1}(Z)$ is exactly the inverse image under f if the topological subspace which underlies Z. Note that $\mathscr{I}\mathscr{O}_X$ can also be described as the kernel of the natural surjective map $\mathscr{O}_X = f^*\mathscr{O}_Y \twoheadrightarrow f^*\mathscr{O}_Z$ obtained by applying the right exact functor f^* to the natural surjective map $\mathscr{O}_Y \twoheadrightarrow \mathscr{O}_Z$

Problems

(1) Let $f\colon X \to Y$ be a map of schemes with Y affine, and $\mathscr{F}$ a quasi-coherent sheaf on X. Show that $\Gamma(Y, \mathbf{R}^i f_*\mathscr{F}) = \mathrm{H}^i(X, \mathscr{F})$ for every $i \geq 0$ and that $\mathrm{H}^i(X, \mathscr{F})^\sim = \mathbf{R}^i f_*\mathscr{F}$. Here functorial isomorphisms are being treated as equalities.

(2) Show that a map of schemes $f\colon X \to Y$ is affine if and only if

$$\mathbf{R}^i f_*\mathscr{F} = 0 \qquad (i \geq 1)$$

for every quasi-coherent sheaf $\mathscr{F}$ on X.

(3) Let $f\colon X \to Y$ be affine. Show that f is an isomorphism if and only if $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is an isomorphism of $\mathscr{O}_Y$-algebras.

(4) Consider the cartesian diagram (†) above, and assume u is *faithfully flat.*

(a) Show that f is affine if and only if f' is.

(b) Show that if f' is an isomorphism then so is f. (The converse is obvious and you have not been asked that.) [*Hint:* Show that f is affine, and then use one of the problems above to see it is an isomorphism. You may also need the facts mentioned about higher direct images and flat base change.]

(5) Suppose S is a scheme. Let $\mathfrak{M}$ be a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which are stable under compositions, base changes, and which contains all isomorphisms. Let F be a **Sets**-valued pre sheaf on $\mathbb{S}\mathrm{ch}_{/S}$,

$$F\colon \left(\mathbb{S}\mathrm{ch}_{/S}\right)^{\circ} \to \mathbf{Sets}$$

such that

$$F(\emptyset) = \bigstar,$$

where $\bigstar$ our designated terminal object in **Sets**. Define a topology $\tau_{\mathfrak{M}}$ on $\mathbb{S}\mathrm{ch}_{/S}$ by decreeing that $\{U_i \to U\}$ is a cover if and only if the maps in the collection are jointly in $\mathfrak{M}$. Show that F is an $\mathfrak{M}$-sheaf if and only if it is a sheaf for the $\tau_{\mathfrak{M}}$ topology on $\mathbb{S}\mathrm{ch}_{/S}$.

(6) Let $f\colon X \to Y$ be a surjective map of schemes. Show that the following are equivalent:

(a) Every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

(b) There exists a covering (in the classical sense of the term) $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.

(c) The map f is surjective and given a point $x \in X$, there exists an open neighborhood U of x in X such that $f(U)$ us open in Y, and the map $U \to f(U)$ induced by f (by restricting to U) is a quasi-compact map.

(d) The map f is surjective and given a point $x \in X$, there exists a quasi-compact open neighborhood U of x in X such that $f(U)$ is open and affine in Y.

(7) Argue using (6) that descent for quasi-coherent sheaves works for fpqc-maps (not only for faithfully flat and quasi-compact maps). Do not reprove descent for faithfully flat quasi-compact maps. You may assume it.

(8) Let S be a scheme and let

$$F\colon \left(\mathbb{S}\mathrm{ch}_{/S}\right)^{\circ} \to \mathbf{Sets}$$

be a presheaf on $\mathbb{S}\mathrm{ch}_{/S}$ such that $F(\emptyset) = \bigstar$. Let $\mathfrak{M}'$ be the class of *faithfully flat and quasi-compact* maps, $\mathfrak{M}''$ the class of maps $V \to U$ such that $V = \coprod_i U_i$ with each U_i an open subscheme of U and $V \to U$ the natural map (open immersion on each U_i) and such that $U = \cup_i U_i$. Let $\mathfrak{M}$ be the class of fpqc-maps. Show that the following are equivalent:

(a) F is an $\mathfrak{M}'$-sheaf and an $\mathfrak{M}''$-sheaf.

(b) F is an $\mathfrak{M}$-sheaf.

[*Hint:* Use Problem (6).]

(9) Let S be a scheme. Consider the cartesian square in $\mathbb{S}\mathrm{ch}_{/S}$

$$(\ddagger) \qquad \begin{array}{ccc} T'' & \xrightarrow{p_2} & T' \\ {\scriptstyle p_1}\downarrow & \square & \downarrow{\scriptstyle p} \\ T' & \xrightarrow[p]{} & T \end{array}$$

with $p\colon T' \to T$ fpqc.

(a) Let $\mathscr{G}$ be a quasi coherent sheaf on T. Suppose we have a surjective map of quasi-coherent $\mathscr{O}_{T'}$-modules

$$\theta\colon p^*\mathscr{G} \twoheadrightarrow \mathscr{F}$$

where $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_{T'}$-module such that there is an isomorphism

$$\varphi\colon p_2^*\mathscr{F} \xrightarrow{\sim} p_1^*\mathscr{F}$$

satisfying the equation $\varphi \circ p_2^*(\theta) = p_1^*(\theta)$ under the identification $p_1^*p^*\mathscr{G} = p_2^*p^*\mathscr{G}$. Show that $(\mathscr{F}, \varphi)$ is a descent datum.

(b) Let $Z' \hookrightarrow T'$ be a closed subschema of T' such that $p_1^{-1}(Z') = p_2^{-1}(Z')$ as closed subschemes of T''. Show that there is a unique closed subschema $Z \hookrightarrow T$ such that $p^{-1}(Z) = Z'$.

(10) Let S be a scheme. In $\mathbb{S}\mathrm{ch}_{/S}$ consider the cartesian diagram ($\ddagger$) above. with $p\colon T' \to T$ fpqc.

(a) Let $f'\colon T' \to Z$ be a map in $\mathbb{S}\mathrm{ch}_{/S}$ such that $f' \circ p_1 = f' \circ p_2$. Show that there exists a unique map of schemes $f\colon T \to Z$ such that $f' = f \circ p$. [*Hint*: Use Problem (8) to reduce to the case where p is faithfully flat and quasi-compact. Next reduce to the case where T and T' and Z are affine. Finally use the graph $\Gamma' = \Gamma_{f'} \hookrightarrow T' \times_S Z$ (show it is a closed subscheme of the product scheme!) and make it "descend" to a closed subschema of $T \times_S Z$. And then?]

(b) Conclude that $h_Z = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(-, Z)$ is an fpqc-sheaf on $\mathbb{S}\mathrm{ch}_{/S}$.

Problems on Principal Bundles. In the problems that follow, we work with the classical notion of a topological space (i.e., not with Grothendieck topologies) and all topological spaces and topological groups that occur are Hausdorff. All group actions of a topological group on a topological space will be assumed to be continuous.

We will deal throughout with a topological group G which acts on the right on a topological space Z, and with a map

$$f\colon Z \to X$$

which is G-equivariant for the trivial action of G on X. We call such an $f\colon Z \to X$ a *G-space over X*, and often simply call Z a G-space over X. Set

$$G_X = X \times G$$

and let

$$\pi_X\colon G_X \to X$$

be the first projection. Note that $g(x, g') = (x, gg')$ gives a left action on G_X and $((x, g')g = (x, g'g)$ a right action on G_X. The space G_X with its right action is clearly G-space over X.

We say $f\colon Z \to X$ is a *trivial* G-space over X if there is a G-equivariant isomorphism (for the right G-action on G_X)

$$\theta\colon G_X \xrightarrow{\sim} Z$$

such that

$$\theta \circ f = \pi_X.$$

Clearly if Z trivial G-space over X then it is a principal bundle over X, in fact the trivial principal bundle.

(11) Let $u\colon W \to X$ be a continuous map and set $Z_W := Z \times_X W$. Let $f_W\colon Z_W \to W$ and $v\colon Z_W \to Z$ be the natural maps. Show that G acts naturally on the right on Z_W in such a way that it is a G-space over W and such that v is G- equivariant.

(12) Suppose $\mathscr{U} = \{U_\alpha\}$ is an open cover of X, and Z_{U_α} is a trivial G-space over U_α for each α. Show that $f\colon Z \to X$ has the natural structure of a principal G-bundle such that the right G-action on Z induced by the principal bundle is the given G-action on Z.

(13) Suppose the G-action on Z is the one induced by a principal G-bundle structure on $f\colon Z \to X$. Let $u\colon W \to X$ be a continuous map. Show that $f_W\colon Z_W \to W$ has a natural structure of a principal G-bundle such that the resulting right G-action on Z_W is the same as the one induced by the right G-action on Z as in Problem (11).

(14) Consider $\mathscr{Z} := Z_Z = Z \times_X Z$, and the induced map $f_Z\colon \mathscr{Z} \to Z$. Define

$$\Psi\colon G_Z \to \mathscr{Z}$$

by $(z,\, g) \mapsto (z,\, zg)$, $z \in Z$, $g \in G$.

(a) Show that Ψ is G-equivariant for the right G-actions on both spaces.

(b) Show that $f_Z \circ \Psi = \pi_Z$.

(c) Show that if $f\colon Z \to X$ is a principal G-bundle (such that the induced right G-action is the given one) then Ψ is an isomorphism.

(d) Suppose $f\colon Z \to X$ has *local sections*, i.e., around each point $x \in X$ there is an open neighborhood such that the restriction $f^{-1}U_x \to U_x$ of f has a section. Suppose further that Ψ is an isomorphism. Show that $f\colon Z \to X$ is a principal bundle and the right G-action on Z induced by its principal bundle structure is the given right G-action on it.