LECTURE 9

1. **Principal Bundles**

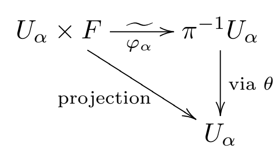
1.1. **Basic assumptions.** We fix once and for all in this and the next lecture a topological Group G. We will assume all topological spaces that occur in these two lectures (including G) are Hausdorff. All groups actions of G on topological spaces that occur will be assumed to be continuous. If A and B are topological spaces, then the symbol $f\colon A \xrightarrow{\sim} B$ means that $f\colon A \to B$ is a homeomorphism. If we do not wish to label the homeomorphism, we simple write $A \xrightarrow{\sim} B$ to denote a specific (though nameless) homeomorphism. If $\mathscr{U} = \{U_\alpha\}$ is an open cover of a topological space X, then

$$U_{\alpha_1 \dots \alpha_n} := U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}.$$

1.2. **Fibre Bundles.** Let F be a topological space. Recall that a continuous map $\pi\colon \mathscr{F} \to X$ between topological spaces is said to be *a fibre bundle with fibre* F if there is an open cover $\mathscr{U} = \{U_\alpha\}$ of X and homeomorphisms (one for each index α)

$$\varphi_\alpha\colon U_\alpha \times F \xrightarrow{\sim} \pi^{-1}U_\alpha$$

such that



commutes. in the above we call X the *base* of the fibre bundle and F the *fibre* of the fibre bundle.

In this case, if

$$\varphi_{\alpha\beta}\colon U_{\alpha\beta} \times F \xrightarrow{\sim} U_{\alpha\beta} \times F$$

denotes the automorphism $\varphi_\alpha^{-1}|_{U_{\alpha\beta}} \circ \varphi_\beta|_{U_{\alpha\beta}}$, then we have maps $h_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{Aut}(F)$ such that

$$\varphi_{\alpha\beta}(u,\, f) = (u,\, h_{\alpha\beta}(u)f) \qquad (u \in U_{\alpha\beta}, f \in F).$$

Here $\mathrm{Aut}(F)$ denotes the group of topological automorphisms of F.

Now suppose G acts on F from the left. We have a natural group homomorphism $G \to \mathrm{Aut}(F)$. We say the above fibre bundle $\pi\colon \mathscr{F} \to X$ *has structure group* G if maps $h_{\alpha\beta}$ above factor through G and the resulting maps $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ are continuous. Note that in this case

$$\varphi_{\alpha\beta}(u,\, f) = (u,\, g_{\alpha\beta}(u)f) \qquad (u \in U_{\alpha\beta}, f \in F).$$

It is clear that for any three indices α, β, and γ we have the *cocycle rules*:

(1.2.1) $$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$$

on $U_{\alpha\beta\gamma}$.

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Data of the form $(g_{\alpha\beta})_{\alpha\beta}$, with the maps $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ continuous and satisfying (1.2.1) on $U_{\alpha\beta\gamma}$ is called a 1-*cocycle*.

Remark 1.2.2. Conversely, given X, $\mathscr{U} = \{U_{\alpha\beta}\}$ and F as above, a 1-cocycle $(g_{\alpha\beta})_{\alpha\beta}$, gives rise to a fibre bundle $\pi\colon \mathscr{F} \to X$ with fibre F and structure group G such that the 1-cocycle induced by $\mathscr{F}$ is $(g_{\alpha\beta})$. Indeed set

$$\mathscr{F} = \coprod_{\alpha} (U_\alpha \times F)/\sim$$

where for $(u_1, f_1) \in U_\alpha \times F$ and $(u_2, f_2) \in U_\beta \times F$, the relationship $(u_1, f_1) \sim (u_2, f_2)$ holds if and only if $u_1, u_2 \in U_{\alpha\beta}$ and $f_1 = g_{\alpha\beta} f_2$. Moreover the two processes (of obtaining a 1-cocycle for a trivialising data for a fibre bundle and of constructing a fibre bundle from a 1-cocycle) are inverse processes.

Remark 1.2.3. In view of Remark 1.2.2, if $\pi\colon \mathscr{F} \to X$ is a fibre bundle with structure group G and F' is a topological space on which G acts on the left, then $\pi\colon \mathscr{F} \to X$ induces a fibre bundle $\pi'\colon \mathscr{F}' \to X$ with fibre F' and structure group G, trivializing over the same open sets that $\mathscr{F}$ does, and having the same transition functions $g_{\alpha\beta}$. Indeed the 1-cocyle $(g_{\alpha\beta})$ arising from $\mathscr{F} \to X$ can be used to glue the $U_\alpha \times F'$ as we did in Remark 1.2.2.

Definition 1.2.4. A *principal G-bundle* $\pi\colon P \to X$ is a fibre bundle with structure group G with fibre also equal to G, with the natural left action of G on itself.

Proposition 1.2.5. *If $\pi\colon P \to X$ is a principal G-bundle then there is a natural right action of G on P which is free, and whose orbits are the fibres of π. Locally, on a trivializing open subset U if X this right G action on P looks like:*

$$(u, g)g^* = (u, gg^*) \qquad u \in U, \text{ and } g, g^* \in G.$$

Proof. It is clear that this local action commutes with left multiplication by the transition functions $g_{\alpha\beta}$ and hence glues. □

Theorem 1.2.6. *Let $\pi\colon E \to X$ be a continuous G-equivariant map with G acting trivially on X and on the right on E. Then this G action on E arises from a natural principal G-bundle structure on $\pi\colon E \to X$ if and only if we have an open cover $\mathscr{U} = \{U_\alpha\}$ of X and G-equivariant homeomorphisms (for the right G-action on $U_\alpha \times G$)*

$$\varphi_\alpha\colon U_\alpha \times G \xrightarrow{\sim} \pi^{-1}(U_\alpha),$$

one for each α, such that

$$\begin{array}{ccc} U_\alpha \times G & \xrightarrow[\varphi_\alpha]{\sim} & \pi^{-1}U_\alpha \\ & \searrow^{\textit{projection}} & \downarrow{\scriptstyle \textit{via } \theta} \\ & & U_\alpha \end{array}$$

commutes for every α.

Proof. This is part of your mid-term. □

1.3. **Examples.**

- If X is a connected, path connected, locally simply connected space and $x_0 \in X$ a fixed point, then the universal cover of X, $\pi\colon \widetilde{X} \to X$ is a principal $\pi(X, x_0)$-bundle
- Suppose $\mathscr{V} \to X$ is a vector bundle of rank n. Then it is afire bundle with fibre $\mathbf{R}^n$ (or $\mathbf{C}^n$) and structure group $GL_n(\mathbf{R})$ (or $GL_n(\mathbf{C})$).

1.4. **Reduction of structure group.** Suppose H is *closed subgroup* of H and $\pi\colon \mathscr{F} \to X$ a fibre bundle with fibre F. Note that H also acts on F. Suppose $\pi\colon \mathscr{F} \to X$ is also a fibre bundle with structure group H. Then we say $\mathscr{F}$ has a reduction of structure group to H. In other words, if we can show that there are trivialisations of $\mathscr{F}$ such that the transition functions $g_{\alpha\beta}$ take values in H, then (and only then) $\mathscr{F} \to X$ has a reduction of structure group to H.