LECTURE 8

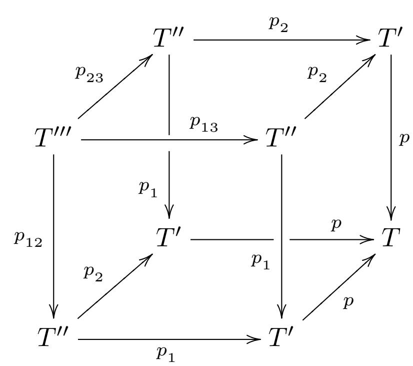
1. Descent for closed subschemes

1.1. **The set theoretic case.** Suppose $p\colon X' \to X$ is a surjective map of sets. Let $X'' := X' \times_X X' = \{(a,b) \in X' \times X' \mid p(a) = p(b)\}$ and for $i = 1, 2$ let $p_i\colon X'' \to X'$, be the two projections. Suppose $Z' \subset X'$ is a subset of X'. One checks that $Z' = p^{-1}(Z)$ for some Z if and only if $p_1^{-1}(Z') = p_2^{-1}(Z')$, i.e., if and only if $Z' \times_X X' = X' \times_X Z'$. Since $p \circ p_1 = p \circ p_2$ the "only if" part is clear. For the converse, suppose $Z' \times_X X' = X' \times_X Z'$. Set $Z = p(Z')$. We claim $Z' = p^{-1}(Z)$. Clearly $Z' \subset p^{-1}(Z)$. To see the reverse inclusion, let $a \in p^{-1}(Z)$. We can find a $b \in Z'$ such that $p(a) = p(b)$. Therefore $(a,\, b) \in X' \times_X Z'$. By our hypothesis, this means $(a,\, b) \in Z' \times_X X'$. In other words, $a \in Z'$. Thus $Z' = p^{-1}(Z')$.

This is the fact that Grothendieck generalized for closed subschemes of schemes.

1.2. **Decent for fpqc maps.** As usual, for any map $T' \to T$ in $\mathbb{S}\mathrm{ch}_{/S}$, T'' and T''' will be given by $T'' := T' \times_T T'$ and $T''' := T' \times_T T' \times_T T'$. The maps $p_1,\, p_2\colon T'' \rightrightarrows T'$ denote the two projections and $p_{12},\, p_{13}\, p_{23}$ the three projections from T''' to T''.

Suppose $p\colon T' \to T$ is fpqc. We have a commutative diagram (with all six faces cartesian):



Since descent (obviously) works for Zariski covers, and we have proved that it works for faithfully flat and quasi-compact maps, therefore it works for fpqc maps (see the October 17 notes for the definition of fpqc maps). We're using the fact that the fpqc topology on $\mathbb{S}\mathrm{ch}_{/S}$ is generated by the Zariski topology and the topology given by faithfully flat and quasi-compact maps. In other words if $\mathscr{F}'$ is a quasi-coherent sheaf on T' and we have an isomorphism $\varphi\colon p_2^*\mathscr{F}' \xrightarrow{\sim} p_1^*\mathscr{F}'$ such that $p_{12}^*(\varphi) \circ p_{23}{}^*(\varphi) = p_{13}^*(\varphi)$, then up to isomorphism, there is a unique quasi-coherent sheaf $\mathscr{F}$ on T satisfying $p^*\mathscr{F} = \mathscr{F}'$.

Exercise: Using the various characterizations of fpqc maps given in the earlier notes, show directly that descent for fpqc maps follows from descent for faithfully flat and quasi-compact maps. [Hint: First reduce to $T = \operatorname{Spec} A$. Next, pick a quasi-compact open subscheme V' of T' such that $p(V') = T$. Then $V' \to T$ is a

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quasi-compact faithfully flat map. Descent works for this, and we have obtain a quasi-coherent sheaf $\mathscr{F}$ on T. To show that the end-product (i.e. $\mathscr{F}$) is independent of the process, consider $V = V' \cup V''$ where V'' is another quasi-compact open subscheme of T' which maps surjectively onto T, and use the fact that V is also quasi-compact, so that descent works for $V \to T$. The uniqueness of the descended sheaf should give you the required result.]

1.3. **Descent for quotient sheaves.** The following two results are part of your Homework.

Proposition 1.3.1. *Let $p\colon T' \to T$ be an* fpqc*-map and $\mathscr{G}$ a quasi-coherent $\mathscr{O}_T$-module. Suppose we have a surjective map of quasi-coherent sheaves $\theta\colon p^*\mathscr{G} \twoheadrightarrow \mathscr{F}$ where $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_{T'}$-module such that there is an isomorphism*

$$\varphi\colon p_2^*\mathscr{F} \xrightarrow{\sim} p_1^*\mathscr{F}$$

satisfying the equation $\varphi \circ p_2^*(\theta) = p_1^*(\theta)$ (under the identification $p_1^*p^*\mathscr{G} = p_2^*p^*\mathscr{G}$). Then $(\mathscr{F}, \varphi)$ is a descent datum. Moreover If $\mathscr{H}$ is the quasi-coherent sheaf on T such that $\mathscr{F} = p^*\mathscr{H}$, then there is a unique surjective map $\gamma\colon \mathscr{G} \twoheadrightarrow \mathscr{H}$ such that $p^*\gamma = \theta$

From the above, one can deduce:

Corollary 1.3.2. [SGA 1, Exposé VIII, Corollaire 1.9] *let $Z' \hookrightarrow T'$ be a closed subscheme of T' such that $p_1^{-1}(Z') = p_2^{-1}(Z')$. Then there is a unique closed subscheme $Z \hookrightarrow T$ such that $p^{-1}(Z) = Z'$.*

Proof. This is part of your HW. □

2. **Schemes are fpqc-sheaves**

Fix a scheme S. In this section we will prove that a scheme X over S is necessarily an fpqc-sheaf on $\mathbb{S}\mathrm{ch}_{/S}$. More precisely, we will prove that h_X is an fpqc-sheaf over $\mathbb{S}\mathrm{ch}_{/S}$. Note that since the fpqc topology is finer than the Zariski, étale, and fppf topologies on $\mathbb{S}\mathrm{ch}_{/S}$, it follows that X is a Zariski, étale, and an fppf-sheaf.

2.1. **The problem restated.** Fix $X \in \mathbb{S}\mathrm{ch}_{/S}$. Suppose

$$\begin{array}{ccc} T'' & \xrightarrow{p_2} & T' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ T' & \xrightarrow[p]{} & T \end{array}$$

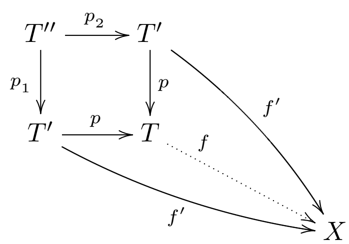
is a cartesian diagram with p (and hence p_1 and p_2) fpqc. In order to show that h_X is an fpqc-sheaf we have to show that the sequence of sets

$$h_X(T) \to h_X(T') \rightrightarrows h_X(T'')$$

is exact, where the first arrow is p^* and the double arrow arises from p_1^* and p_2^*.

The problem can be rephrased as follows. Suppose $f'\colon T' \to X$ is a map in $\mathbb{S}\mathrm{ch}_{/S}$ such that $f' \circ p_1 = f' \circ p_2$. Then there is a unique map $f\colon T \to X$ in $\mathbb{S}\mathrm{ch}_{/S}$ such that $f' = f \circ p$. In other words, if the diagram of solid arrows below commutes, then the dotted arrow can be filled in a unique way to make the whole diagram commute.

(2.1.1)



Proof. Part of your HW. □

We summarize the above in the form of the following theorem:

Theorem 2.1.2. *Let X be an S-scheme. Then X is a sheaf on the* fpqc *site (whence on the* Zariski, étale*, and* fppf *sites) on* $\mathbb{S}ch_{/S}$*.*

We point out that the hierarchy of topologies on $\mathbb{S}\mathrm{ch}_{/S}$, with the arrows pointing toward finer topologies, is:

$$\text{Zariski} \to \text{étale} \to \text{fppf} \to \text{fpqc}.$$

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