LECTURE 7

1. Examples

In what follows, S is a scheme and $\mathbb{S}\mathrm{ch}_{/S}$ denotes the category of S-schemes, i.e., the category whose objects are maps of schemes $X \to S$ and whose morphisms are commutative diagrams:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

If $X \to S$ is an object in $\mathbb{S}\mathrm{ch}_{/S}$, we often think of X itself as the object, if the underlying map (the so called *structural map* or sometimes the *structure map*) from X to S is understood. From this point of view, a morphism in $\mathbb{S}\mathrm{ch}_{/S}$ is simply a map of the underlying schemes which is compatible with the structure maps of the source and target.

Let $\mathbf{P}$ be a property of maps in $\mathbb{S}\mathrm{ch}_{/S}$ (e.g, $\mathbf{P}$= faithfully flat). A set of maps $\{T_i \to T\}$ in $\mathbb{S}\mathrm{ch}_{/S}$ is said to be *jointly* $\mathbf{P}$ if the induced map of schemes

$$\coprod_i T_i \to T$$

has property $\mathbf{P}$.

Here are some examples of Grothendieck topologies on $\mathbb{S}\mathrm{ch}_{/S}$. There is some confusion regarding terminlogy in 4), 5), 6) and 7) (see Remark 1.1 below).

1) *The big Zariski site.* A covering $\{U_i \to U\}$ is a collection of open immersions which are jointly surjective.

2) *The big étale site.* A covering $\{U_i \to U\}$ is a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which is jointly surjective and étale.

3) *The faithfully flat site.* A covering $\{U_i \to U\}$ is a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which is jointly faithfully flat.

4) *The faithfully flat and quasi compact site.* A covering $\{U_i \to U\}$ is a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which is jointly faithfully flat and quasi-compact ($\mathbf{P}$ = faithully flat and quasi-compact).

5) *The fpqc site.* The abbereviation fpqc is for "fidèlement plat et quasi-compact". In the original definition in SGA fpqc meant exactly what it says, namely, a map is fpqc if it is faithfully flat and quasi-compact. One problem with this is that a Zariski cover—if consisting of a disjoint union of infinite open immersions—need not be quasi-compact. Following Kleiman's suggestion, Vistoli has the following definition [FGA-ICTP, Def. 2.34, p. 28]. We say that a map of schemes $f\colon X \to Y$ is fpqc if it is faithfully flat (or simply flat) and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X. One checks (this may be asked in your HW or mid-term take home) that $f\colon X \to Y$ being fpqc is equivalent to any of the following conditions:

Date: September 5, 2012.

(1) The map f is faithfully flat (or simply flat) and there exists a covering (in the classical sense of the term) $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.
(2) The map f is faithfully flat and given a point $x \in X$, there exists an open neighborhood U of x in X such that $f(U)$ us open in Y, and the restriction $U \to f(U)$ induced by f is a quasi-compact map.
(3) The map f is faithfully flat and given a point $x \in X$, there exists an open neighborhood U if x in X such that $f(U)$ is open and affine in Y.

6) *The faithfully flat and finitely presented site.* Coverings are jointly faithfully flat and finitely presented, a notion which we now define. Recall that a map of rings $\varphi\colon A \to B$ is called finitely presented if B is generated as an A-algebra by a finite number of elements, and the ideal relations between the generators is finitely generated. In other words, φ factors as

$$A \to A[X_1, \ldots, X_n] \twoheadrightarrow B.$$

for some n (the "two-headed" arrow denotes a set-theoretic surjection) and the kernel $A[X_1, \ldots, X_n] \twoheadrightarrow B$ is a finitely generated ideal. One can therefore make sense of *locally finitely presented* maps of schemes. A map of schemes $X \to Y$ is said to be *finitely presented* if it is locally finitely presented, quasi compact and if it is *quasi-separated*, i.e., if its diagonal morphism $\delta\colon X \to X \times_Y X$ is also quasi-compact. One checks that the two notions of finite presentation on maps of affine schemes (one coming from the algebra definition and the other from the schemes definition) coincide. For this one notes that a map of affine schemes is always quasi-compact and quasi-separated. This reduces the problem to showing that local finite presentation and finite presentation are equivalent concepts for a map of rings $A \to B$. See [EGA, Corollaire (6.3.9), p. 306] for details.

7) *The fppf site.* The abbreviation fppf is for "fidèlement plat et de présentation finie". Coverings $\{U_i \to U\}$ are jointly faithfully flat and locally of finite presentation. [FGA-ICTP, Example 2.32, p. 27].

Remark 1.1. Confusingly, Example 4) above is often called the fpqc-site and Example 6) the fppf-site. This however doesn't take into account compatibility issues with Zariski coverings (which need not be jointly quasi-compact). I will follow Kleiman and Vistoli's lead in these matters, and reserve fpqc and fppf for 5) and 7) above respectively. Note that using the definitions in 5) and 7) a Zariski covering is a legitimate covering in the fpqc and in the fppf topology. In other words fppf and fpqc (as we have defined them) are finer topologies than the Zariski topology, but this is not so for the topologies defined by 4) and 6) above. The traditional way of getting around this to make a distinction between sheaves on the fpqc (resp. fppf) topology and fpqc-sheaves (resp. fppf-sheaves). In greater detail, let the topology defined in 4) (resp. 6)) above be called fpqc^1 (resp. fppf^1). Then, according to [BLR, § 8.1], a presheaf F on $\mathbb{S}\text{ch}_{/S}$ is an fpqc-sheaf (resp. fppf-sheaf) if it is an fpqc^1-sheaf as well as a Zariski-sheaf (resp. an fppf^1-sheaf as well as a Zariski sheaf). Since it is not hard to see that the topology generated by fpqc^1 and the Zariski topology is the fpqc-topology, and the analogous statament for the fppf situation is also easy to see, one notes that F is an fppf-sheaf (resp. fppf-sheaf) by our definition if and only if it so by the definition in [BLR]. Compare with [BLR, § 8.1, pp. 199—201]

2. Sheaves without topology - an alternate approach

Definition 2.1. Let $\mathfrak{M}$ be a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which are stable under compositions, base change, and which contains all isomorphisms. An $\mathfrak{M}$-sheaf is a functor (i.e. a presheaf)

$$F\colon \left(\mathbb{S}\mathrm{ch}_{/S}\right)^{\circ} \to (\mathrm{Sets})$$

such that

(1) $F(\coprod_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$. (Presheaves satisfying this condition are called *prepared presheaves.*)

(2) Given $T' \to T$ in $\mathfrak{M}$, with $T'' := T' \times_T T'$, $p_i\colon T'' \to T'$, $i = 1, 2$ the projections, the sequence of sets

$$F(T) \to F(T') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} F(T'')$$

is exact. (Here, as usual, $p_i^* := F(p_i)$, $i = 1, 2$.)

The exactness of the sequence of sets in (2) above means that if an element $\xi \in F(T')$ satisfies the equation $p_1^*(\xi) = p_2^*(\xi)$ then there is unique element $\zeta \in F(T)$ such that ξ is the pullback of ζ under $T' \to T$.

2.2. **Examples.** All collections below are collections of maps in $\mathbb{S}\mathrm{ch}_{/S}$.

(1) $\mathfrak{M}_{\mathrm{Zar}}$ consists of surjective maps $T' \to T$ such that $T' = \coprod_\alpha T_\alpha$ and the restriction $T_\alpha \to T$ of $T' \to T$ is an open immersion for each α.
(2) $\mathfrak{M}_{\text{ét}}$ consists of étale surjective maps.
(3) $\mathfrak{M}_{\mathrm{fppc}}$ consists of fppf maps.
(4) $\mathfrak{M}_{\mathrm{fpqc}}$ consists of fpqc maps.

There are two advantages to this approach, namely (a) one does not have to define a Grothendieck topology on $\mathbb{S}\mathrm{ch}_{/S}$, and yet can do sheaf theory, and (b) coverings occur only in the form of a single map in $\mathfrak{M}$.

Remark 2.2.1. $\mathfrak{M}$ as above gives a topology on $\mathbb{S}\mathrm{ch}_{/S}$, namely the topology such that $\{U_i \to U\}$ is a covering if and only if it is jointly in $\mathfrak{M}$, i.e., $\coprod_i U_i \to U$ is in $\mathfrak{M}$.

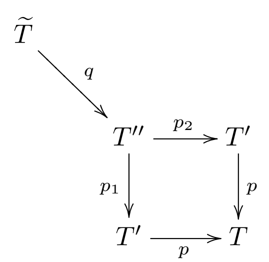
3. Sheafification in the $\mathfrak{M}$-topology for prepared presheaves

For $p\colon T' \to T$ in $\mathfrak{M}$, the symbols T'', p_1, p_2 etc will have their usual meaning. Let

$$F\colon \left(\mathbb{S}\mathrm{ch}_{/S}\right)^{\circ} \to (\mathrm{Sets})$$

be a functor such that $F(\coprod_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$.

Let $T \in \mathbb{S}\mathrm{ch}_{/S}$. Consider pairs $\theta = (T' \xrightarrow{p} T, \xi)$ such that $p\colon T' \to T$ is in $\mathfrak{M}$, $\xi \in F(T')$, which satisfy the property that there exists a map $q\colon \widetilde{T} \to T''$ in $\mathfrak{M}$ such that $q^* p_1^* \xi = q^* p_2^* \xi$.



Suppose $\theta_1 = (T_1 \to T,\, \xi_1)$ and $\theta_2 = (T_2 \to T,\, \xi_2)$ are two such pairs. We say θ_1 is equivalent to θ_2, if there exists a map $\widetilde{T} \to T_1 \times_T T_2$ in $\mathfrak{M}$ such that the pull backs of ξ_1 and ξ_2 to $\widetilde{T}$ are the same under the two composites:

$$\widetilde{T} \to T_1 \times_T T_2 \to T_1$$

$$\widetilde{T} \to T_1 \times_T T_2 \to T_2.$$

Define $F^+(T)$ to be the "set" of "equivalence classes" of such pairs $\theta = (T' \to T,\, \xi)$. The issues of logic that crop up are solved by fixing universes. However, the answers depend on the universe so fixed. Thankfully this poses no problem for the fppf, étale, and Zariski topologies on $\mathbb{S}\mathrm{ch}_{/S}$.

Clearly, if $\xi \in F(T)$ then $\theta_\xi = (T \xrightarrow{1_T} T,\, \xi)$ is a pair of the kind being considered. Moreover $\theta' = (T' \xrightarrow{p} T,\, \xi')$ is equivalent to θ if and only if $q \circ \xi' = q \circ p^*\xi$ for a map $q\colon \widetilde{T} \to T'$ in $\mathfrak{M}$. We thus have an obvious map of presheaves $u_F\colon F \to F^+$. It is easy to check that if G is an $\mathfrak{M}$-sheaf on $\mathbb{S}\mathrm{ch}_{/S}$ and $\varphi\colon F \to G$ is a map of pre sheaves, then there exists a unique map $\varphi^+\colon F^+ \to G$ such that $\varphi^+ \circ u_F = \varphi$.

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