LECTURE 5

1. Descent for Schemes

1.1. Suppose we have a cartesian square

$$\begin{array}{ccc} X'' & \xrightarrow{p_2} & X' \\ {\scriptstyle p_1}\downarrow & \square & \downarrow{\scriptstyle p} \\ X' & \xrightarrow[p]{} & X \end{array}$$

with $p\colon X' \to X$ *faithfully flat and quasi-compact*. Let $Z' \xrightarrow{g} X'$ be an X' scheme. For $i = 1, 2$, let $p_i^* Z'$ be the base change of Z' via the map $p_i\colon X'' \to X'$. An isomorphism

$$\varphi\colon p_2^* Z' \xrightarrow{\sim} p_1^* Z'$$

is said to be a descent datum on the X'-scheme Z' if

$$p_{12}^* \varphi \circ p_{23}^* \varphi = p_{13}^* \varphi$$

In this case (Z', φ) is said to be a descent datum on $X' \xrightarrow{p} X$.

We say that (Z', φ) is an *effective* descent datum if φ is an descent datum and there is a unique (up to isomorphism) X-scheme Z such that $p^* Z \xrightarrow{\sim} Z'$ and the natural descent datum φ_Z on $p^* Z$ corresponds to φ under this isomorphism.

Lemma 1.1.1. *If $Z' \xrightarrow{g} X'$ is an affine map and φ is a descent datam on $Z' \to X$, then it is effective*

Proof. Let $\mathscr{A}' = g_* \mathscr{O}_{Z'}$. Then $\mathscr{A}'$ is a sheaf of $\mathscr{O}_{X'}$-algebras with descent datum coming from φ. So $\mathscr{A}'$ descends to an $\mathscr{O}_X$-algebra $\mathscr{A}$. Set $X = \mathbf{Spec}\, \mathscr{A}$. □

2. Topologies and sites

2.1. Definitions. We give the most common definition of a Grothendieck topology. There are other definitions, which use the notion of a *sieve*. However, we wish to traverse a geodesic path to Picard schemes, and for that the usual definition is adequate. And it is the one closest to our intuition.

Definition 2.1.1. Let $\mathscr{C}$ be a category. A *Grothendieck topology* on $\mathscr{C}$ is an assignment, for each object U of $\mathscr{C}$, of a collection of sets of arrows[1] $\{U_i \to U\}$ called *coverings* (of U) such that:

(1) If $V \to U$ is an isomorphism in $\mathscr{C}$, then the singleton set $\{V \to U\}$ is a covering of U.
(2) If $\{U_i \to U\}$ is a covering and $V \to U$ is an arrow in $\mathscr{C}$, then the fiber products $V \times_U U_i$ exist in $\mathscr{C}$, and the set of projections $\{V \times_U U_i \to V\}$ is also a covering.

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[1]arrows = morphisms

(3) If $\{U_i \to U\}$ is a covering and for each i, $\{U_{ij} \to U_i\}$ is a covering of U_i, then the set of composites $\{U_{ij} \to U_i \to U\}$ (as i and j vary) is a covering of U.

A *site* is a category $\mathscr{C}$ together with a Grothendieck topology on it.

The standard example is that of a topological space X. Let $\widehat{X}$ be the category whose objects are the open sets of X and whose morphisms are given by

$$\mathrm{Hom}_{\widehat{X}}(U, V) = \begin{cases} \emptyset \text{ if } U \nsubseteq V \\ U \subseteq V \text{ otherwise} \end{cases}$$

for two objects U and V in $\widehat{X}$. For U an object in $\widehat{X}$ a covering is a collection $\{U_\alpha \to U\}$ where the U_α give a covering (in the usual, set theoretic, sense) of U. Note that in this case each U_α is an open subset of U, whence $\{U_\alpha\}$ is an open covering (in the usual classical sense) of U. One checks easily that this notion of coverings defines a Grothendieck topology on $\widehat{X}$. Indeed, if $U \in \widehat{X}$, then the only isomorphism in $\widehat{X}$ with target U is the identity map, and this is clearly a covering. Next note that if U and U' are open subsets of V (V an open subset of X), then $U \times_V U'$ exists. In fact $U \times_V U' = U \cap U'$. From this observation, (2) is immediate. The third axiom is equally trivial to verify.

We will give other examples later.

3. Sheaf theory

3.1. The classical case. Recall that a presheaf F of sets on a topological space X is an assignment of a set $F(U)$ for each open set U of X, together with "restriction maps" $\rho^U_V \colon F(U) \to F(V)$ for every pair of open sets U and V with $U \supset V$, these restriction maps satisfying $\rho^V_W \circ \rho^U_V = \rho^U_W$ for every pair of inclusion $W \subset V \subset U$. A little thought shows that F is then a contravariant (Sets)-valued functor on $\widehat{X}$. Conversely, a contravariant (Sets)-valued functor F on $\widehat{X}$ is a presheaf, with $\rho^U_V = F(V \subseteq U)$. Note that open coverings play no role in defining pre-sheaves. In other words, if F is a presheaf on X, the category $\widehat{X}$ certainly comes into play for F, but not the Grothendieck topology on $\widehat{X}$. In other words the specific site which is overlaid on $\widehat{X}$ is unimportant for F.

Let F be a presheaf of sets on X. For simplicity we write $\rho^U_V(s) = s|_V$ for $s \in F(U)$ and call the common value the "restriction" of s to V or—with a view toward the general definition of a sheaf on a site—the "pullback" of s to V. For an open set U, the elements of $F(U)$ are often called sections of F over U (or simply sections over U, or just sections). Recall that F is a sheaf if the following two conditions hold:

- If s, t are two elements of $F(U)$, and $\{U_\alpha\}$ is an open covering of U such that $s|_{U_\alpha} = t|_{U_\alpha}$ for every α, then $s = t$.
- If $\{U_\alpha\}$ is an open covering of an open set U of X, and for each index α we have a section $s_\alpha \in F(U_\alpha)$ such that for every pair of indices α and β, $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, then we have a unique section $s \in F(U)$ such that $s_\alpha = s|_{U_\alpha}$.

3.2. Sheaves on sites. Presheaves and sheaves can be defined on sites. In direct analogy with the classical case we make the following definition.

Definition 3.2.1. A *presheaf* on a site $\mathscr{C}$ is a contravariant (Sets)-valued functor on $\mathscr{C}$. Let F be a presheaf on $\mathscr{C}$.

(1) If $f\colon U \to V$ is an arrow in $\mathscr{C}$, s an element of $F(V)$, then the image of s in $F(U)$ under the map $F(f)\colon F(V) \to F(U)$ is called the *pullback* of s under f and is often denoted $f^*(s)$ (rather than $F(f)(s)$). For any $U \in \mathscr{C}$, an element of $F(U)$ is called *a section of F on U*.

(2) F is said to be *separated* if given a covering $\{U_i \to U\}$ and two sections a and b in $F(U)$ such that $a_i = b_i$ for every i, where a_i and b_i are the pullbacks of a and b respectively to U_i, then $a = b$.

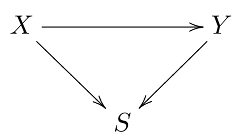
(3) F is said to be a *sheaf* on $\mathscr{C}$ if it satisfies the following condition for every covering $\{U_i \to U\}$ in $\mathscr{C}$:

For every pair of indices i and j, let $\mathrm{pr}_1\colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_U U_j \to U$ denote the two projections. If there are sections $a_i \in F(U_i)$—one for each index i—such that $\mathrm{pr}_1^*(a_i) = \mathrm{pr}_2^*(a_j)$ for every i and j, then there exists a unique section $a \in F(U)$ such that the pull back of a to U_i is a_i for every i.

Note that the notion of a presheaf requires only the category underlying the site $\mathscr{C}$ and not the topology on $\mathscr{C}$. However, the notion of a sheaf requires knowing the site (in particular the topology on $\mathscr{C}$). Note also that a sheaf is necessarily a separated presheaf.

4. Examples

In what follows, S is a scheme and $\mathbb{S}\mathrm{ch}_{/S}$ denotes the category of S-schemes, i.e., the category whose objects are maps of schemes $X \to S$ and whose morphisms are commutative diagrams:



If $X \to S$ is an object in $\mathbb{S}\mathrm{ch}_{/S}$, we often think of X itself as the object, if the underlying map (the so called *structural map* or sometimes the *structure map*) from X to S is understood. From this point of view, a morphism in $\mathbb{S}\mathrm{ch}_{/S}$ is simply a map of the underlying schemes which is compatible with the structure maps of the source and target.

Let $\mathbf{P}$ be a property of maps in $\mathbb{S}\mathrm{ch}_{/S}$ (e.g, $\mathbf{P}$= faithfully flat). A set of maps $\{T_i \to T\}$ in $\mathbb{S}\mathrm{ch}_{/S}$ is said to be *jointly* $\mathbf{P}$ if the induced map of schemes

$$\coprod_i T_i \to T$$

has property $\mathbf{P}$.

Here are some examples of Grothendieck topologies on $\mathbb{S}\mathrm{ch}_{/S}$. There is some confusion regarding terminlogy in 4), 5), 6) and 7) (see Remark 4.1 below).

1) *The big Zariski site.* A covering $\{U_i \to U\}$ is a collection of open immersions which are jointly surjective.

2) *The big étale site.* A covering $\{U_i \to U\}$ is a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which is jointly surjective and étale.

3) *The faithfully flat site.* A covering $\{U_i \to U\}$ is a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which is jointly faithfully flat.

4) *The faithfully flat and quasi compact site.* A covering $\{U_i \to U\}$ is a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which is jointly faithfully flat and quasi-compact ($\mathbf{P}$ = faithully flat and quasi-compact).

5) *The fpqc site.* The abbereviation fpqc is for "fidèlement plat et quasi-compact". In the original definition in SGA fpqc meant exactly what it says, namely, a map is fpqc if it is faithfully flat and quasi-compact. One problem with this is that a Zariski cover—if consisting of a disjoint union of infinite open immersions—need not be quasi-compact. Following Kleiman's suggestion, Vistoli has the following definition [FGA-ICTP, Def. 2.34, p. 28]. We say that a map of schemes $f\colon X \to Y$ is fpqc if it is faithfully flat (or simply flat) and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X. One checks (this may be asked in your HW or mid-term take home) that $f\colon X \to Y$ being fpqc is equivalent to any of the following conditions:

(1) The map f is faithfully flat (or simply flat) and there exists a covering (in the classical sense of the term) $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.
(2) The map f is faithfully flat and given a point $x \in X$, there exists an open neighborhood U of x in X such that $f(U)$ us open in Y, and the restriction $U \to f(U)$ induced by f is a quasi-compact map.
(3) The map f is faithfully flat and given a point $x \in X$, there exists an open neighborhood U if x in X such that $f(U)$ is open and affine in Y.

6) *The faithfully flat and finitely presented site.* Coverings are jointly faithfully flat and finitely presented, a notion which we now define. Recall that a map of rings $\varphi\colon A \to B$ is called finitely presented if B is generated as an A-algebra by a finite number of elements, and the ideal relations between the generators is finitely generated. In other words, φ factors as

$$A \to A[X_1, \dots, X_n] \twoheadrightarrow B.$$

for some n (the "two-headed" arrow denotes a set-theoretic surjection) and the kernel $A[X_1, \dots, X_n] \twoheadrightarrow B$ is a finitely generated ideal. One can therefore make sense of *locally finitely presented* maps of schemes. A map of schemes $X \to Y$ is said to be *finitely presented* if it is locally finitely presented, quasi compact and if it is *quasi-separated*, i.e., if its diagonal morphism $\delta\colon X \to X \times_Y X$ is also quasi-compact. One checks that the two notions of finite presentation on maps of affine schemes (one coming from the algebra definition and the other from the schemes definition) coincide. For this one notes that a map of affine schemes is always quasi-compact and quasi-separated. This reduces the problem to showing that local finite presentation and finite presentation are equivalent concepts for a map of rings $A \to B$. See [EGA, Corollaire (6.3.9), p. 306] for details.

7) *The fppf site.* The abbreviation fppf is for "fidèlement plat et de présentation finie". Coverings $\{U_i \to U\}$ are jointly faithfully flat and locally of finite presentation. [FGA-ICTP, Example 2.32, p. 27].

Remark 4.1. Confusingly, Example 4) above is often called the fpqc-site and Example 6) the fppf-site. This however doesn't take into account compatibility issues with Zariski coverings (which need not be jointly quasi-compact). I will follow Kleiman and Vistoli's lead in these matters, and reserve fpqc and fppf for 5) and 7) above respectively. Note that using the definitions in 5) and 7) a Zariski covering is a legitimate covering in the fpqc and in the fppf topology. In other words fppf and fpqc (as we have defined them) are finer topologies than the Zariski topology, but this is not so for the topologies defined by 4) and 6) above. The traditional way of getting around this to make a distinction between sheaves on the fpqc (resp. fppf) topology and fpqc-sheaves (resp. fppf-sheaves). In greater detail, let the topology defined in 4) (resp. 6)) above be called fpqc^1 (resp. fppf^1). Then, according to [BLR, § 8.1], a presheaf F on $\mathbb{S}\mathrm{ch}_{/S}$ is an fpqc-sheaf (resp. fppf-sheaf) if it is an fpqc^1-sheaf as well as a Zariski-sheaf (resp. an fppf^1-sheaf as well as a Zariski sheaf). Since it is not hard to see that the topology generated by fpqc^1 and the Zariski topology is the fpqc-topology, and the analogous statament for the fppf situation is also easy to see, one notes that F is an fppf-sheaf (resp. fppf-sheaf) by our definition if and only if it so by the definition in [BLR]. Compare with [BLR, § 8.1, pp. 199—201]

5. Alternate approach

Definition 5.1. Let $\mathfrak{M}$ be a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which are stable under compositions, fiber products and which contains all isomorphisms. An $\mathfrak{M}$-sheaf is a functor (i.e. a presheaf)

$$F\colon \left(\mathbb{S}\mathrm{ch}_{/S}\right)^{\circ} \to (\mathrm{Sets})$$

such that

(1) $F(\coprod_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$.
(2) Given $T' \to T$ in $\mathfrak{M}$, with $T'' := T' \times_T T'$, $p_i\colon T'' \to T'$, $i = 1, 2$ the projections, the sequence of sets

$$F(T) \to F(T') \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} F(T'')$$

is exact. (Here, as usual, $p_i^* := F(p_i)$, $i = 1, 2$.)

The exactness of the sequence of sets in (2) above means that if an element $\xi \in F(T')$ satisfies the equation $p_1^*(\xi) = p_2^*(\xi)$ then there is unique element $\zeta \in F(T)$ such that ξ is the pullback of ζ under $T' \to T$.

5.2. Examples. All collections below are collections of maps in $\mathbb{S}\mathrm{ch}_{/S}$.

(1) $\mathfrak{M}_{\mathrm{Zar}}$ consists of surjective maps $T' \to T$ such that $T' = \coprod_\alpha T_\alpha$ and the restriction $T_\alpha \to T$ of $T' \to T$ is an open immersion for each α.
(2) $\mathfrak{M}_{\text{ét}}$ consists of étale surjective maps.
(3) $\mathfrak{M}_{\mathrm{fppc}}$ consists of fppf maps.
(4) $\mathfrak{M}_{\mathrm{fpqc}}$ consists of fpqc maps.

There are two advantages to this approach, namely (a) one does not have to define a Grothendieck topology on $\mathbb{S}\mathrm{ch}_{/S}$, and yet can do sheaf theory, and (b) coverings occur only in the form of a single map in $\mathfrak{M}$.

References

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