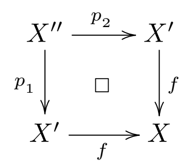
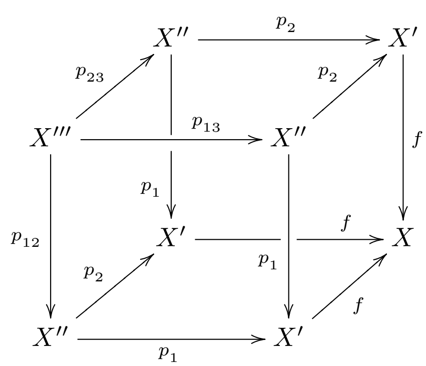
LECTURE 4

1. **Descent for quasi-coherent sheaves on a scheme**

1.1. For any scheme Z, let q-coh$_Z$ denote the category of quasi-cohenrent $\mathscr{O}_Z$-modules. Let $f\colon X' \to X$ be a map of schemes. We have a cartesian square (with $X'' = X' \times_X X'$ and $p_1, p_2\colon X'' \rightrightarrows X$ the two projection):



Setting X''' equal to $X' \times_X X' \times_X X'$ and p_{12}, p_{13}, and p_{23} equal to the obvious three projection maps $X''' \to X''$ given by the formulae $(x_1, x_2, x_3) \mapsto (x_1, x_2)$, $(x_1, x_2, x_3) \mapsto (x_1, x_3)$, and $(x_1, x_2, x_3) \mapsto (x_2, x_3)$ respectively, we have the following three dimensional diagram with each of the six faces being cartesian:



The definitions of the various p_{ij} then imply:

$$p_1 \circ p_{12} = p_1 \circ p_{13}$$
$$p_2 \circ p_{12} = p_1 \circ p_{23}$$
$$p_2 \circ p_{13} = p_2 \circ p_{23}.$$

Definition 1.1.1. Let $\mathscr{G} \in$ q-coh$_{X'}$. A descent datum (on $\mathscr{G}$) with respect to $X' \xrightarrow{f} X$ is an isomorphism

$$\varphi\colon p_2^*\mathscr{G} \xrightarrow{\ \sim\ } p_1^*\mathscr{G}$$

such that the cocycle relation

$$p_{12}^*(\varphi) \circ p_{23}^*(\varphi) = p_{13}^*(\varphi) \tag{1.1.1.1}$$

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holds, i.e., such that the diagram

(1.1.1.2)
$$\begin{array}{ccc} p_{13}^*p_2^*\mathscr{G} & \xrightarrow{p_{13}^*\varphi} & p_{13}^*p_1^*\mathscr{G} \\ \| & & \| \\ p_{23}^*p_2^*\mathscr{G} & & p_{12}^*p_1^*\mathscr{G} \\ {\scriptstyle p_{23}^*\varphi}\downarrow & & \uparrow{\scriptstyle p_{12}^*\varphi} \\ p_{23}^*p_1^*\mathscr{G} & = & p_{12}^*p_2^*\mathscr{G} \end{array}$$

commutes.

One can, in an obvious way, make a category out of quasi-coherent sheaves on X' with descent data withe respect to f. Denote by $\text{q-coh}_{X'\to X}$ (or q-coh_f) the category whose objects are pairs $(\mathscr{G}, \varphi)$, where $\mathscr{G} \in \text{q-coh}_{X'}$ and φ is a descent datum on $\mathscr{G}$. Moreover, with above notations, if $\mathscr{F} \in \text{q-coh}_X$, then the natural isomorphism (in fact an identity by our conventions implicit in (3.1.1.2) above)

$$\varphi^f_{\mathscr{F}}(=\varphi_{\mathscr{F}})\colon p_2^*f^*\mathscr{F} \xrightarrow{\ \sim\ } p_1^*f^*\mathscr{F}$$

is a descent datum on $f^*\mathscr{F}$. Clearly the assignment $\mathscr{F} \mapsto (f^*\mathscr{F}, \varphi_{\mathscr{F}})$ is functorial in $\mathscr{F} \in \text{q-coh}_X$. Let

(1.1.2)
$$F\colon \text{q-coh}_X \to \text{q-coh}_{(X'\to X)}$$

denote the resulting functor. Theorem 2.1.1 can be (obviously) restated as: *Let $f\colon X' \to X$ be a faithfully flat map of* **affine schemes**. *Then the functor F in* (3.1.2) *is an equivalence of categories.*

1.2. To generalize the above statement to situations beyond maps of affine schemes, we need to recall that a scheme is called quasi-compact if every open cover has a finite subcover. Every affine scheme (noetherian or not) is quasi-compact. A map of schemes $f\colon X \to Y$ is said to be quasi-compact if the inverse image of a quasi-compact set is quasi-compact. We will show later in this lecture that the natural transformation $F\colon \text{q-coh}_X \to \text{q-coh}_{(X'\to X)}$ of (3.1.2) is an equivalence of categories whenever $f\colon X' \to X$ is a faithfully flat *quasi-compact* map. The quasi-compactness assumption allows us to reduce to the affine case. As a first step we will now show that F is a fully faithful functor under these hypotheses. This means that if $f\colon X' \to X$ is quasi-compact and faithfully flat, $\mathscr{F}$ and $\mathscr{G}$ quasi-coherent sheaves on X, and

$$\beta\colon (f^*\mathscr{F}, \varphi_{\mathscr{F}}) \to (f^*\mathscr{G}, \varphi_{\mathscr{G}})$$

a map in $\text{q-coh}_{(X'\to X)}$, then there is a unique map $\alpha\colon \mathscr{F} \to \mathscr{G}$ in q-coh_X such that $f^*\alpha = \beta$. The question is local on X by the uniqueness assertion. Therefore, without loss of generality, we may assume that $X = \operatorname{Spec} A$. Quasi-compactness of f then implies that X' can be covered by a finite number of affine open subschemes $X'_\alpha = \operatorname{Spec} B_\alpha$ of X'. Since the collection of indices α is finite, the scheme $\bar{X}'$ given by $\bar{X}' := \coprod_\alpha X'_\alpha$ is affine. In fact $\bar{X}' = \operatorname{Spec}(\prod_\alpha B_\alpha)$. Let $\pi\colon \bar{X}' \to X'$ be the natural map and $\bar{f}\colon \bar{X}' \to X$ the composite $f \circ \pi$. In other words we have a commutative diagram

$$\begin{array}{ccc} \bar{X}' & \xrightarrow{\ \pi\ } & X' \\ & \overset{\bar{f}}{\searrow} & \Big\downarrow{\scriptstyle f} \\ & & X \end{array}$$

and every arrow in the above diagram is a faithfully flat quasi-compact map. To lighten notation we write Z and $\bar{Z}$ for $X' \times_X X'$ and $\bar{X}' \times_X \bar{X}'$ respectively (rather than the more familiar X'' and $\bar{X}''$) and let $p_1, p_2 \colon Z \rightrightarrows X'$, and $\bar{p}_1, \bar{p}_2 \colon \bar{Z} \rightrightarrows \bar{X}'$ be the projections. We then have maps $q \colon Z \to X$ and $\bar{q} \colon \bar{Z} \to X$ given by the compsites $q = f \circ p_1 = f \circ p_2$ and $\bar{q} = \bar{f} \circ \bar{p}_1 = \bar{f} \circ \bar{p}_2$. For $\mathscr{F}, \mathscr{G} \in \text{q-coh}_X$ we have the following commutative diagram.

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F},\, \mathscr{G}) & \xrightarrow{\ f^*\ } & \mathrm{Hom}_{\mathscr{O}_{X'}}(f^*\mathscr{F},\, f^*\mathscr{G}) & \xrightarrow{\ p_1^* - p_2^*\ } & \mathrm{Hom}_{\mathscr{O}_Z}(q^*\mathscr{F},\, q^*\mathscr{G}) \\ & \Big\| & & \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle (\pi\times\pi)^*} \\ 0 \longrightarrow & \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F},\, \mathscr{G}) & \xrightarrow[\ \bar{f}^*\ ]{} & \mathrm{Hom}_{\mathscr{O}_{\bar{X}'}}(\bar{f}^*\mathscr{F},\, \bar{f}^*\mathscr{G}) & \xrightarrow[\ \bar{p}_1^* - \bar{p}_2^*\ ]{} & \mathrm{Hom}_{\mathscr{O}_{\bar{Z}}}(\bar{q}^*\mathscr{F},\, \bar{q}^*\mathscr{G}) \end{array}$$

The bottom row is exact since $\bar{f} \colon \bar{X}' \to X$ is a map of affine schemes, whence Theorem 2.1.1 applies. To show that the functor F of (3.1.2) is fully faithful, we have to show that the top row is also exact. Since π and $\pi \times \pi$ are faithfully flat, the downward arrows are injective (see Remark **??**, esp. towards the end), whence the top row is exact.

Assuming Theorem 2.1.1, we are now in a position to prove:

Theorem 1.2.1. *Let $f \colon X' \to X$ be a faithfully flat quasi-compact map of schemes. Then the functor $F \colon \text{q-coh}_X \to \text{q-coh}_{(X' \to X)}$ of* (3.1.2) *is an equivalence of categories.*

Proof. The question is local on X as can be checked (exercise). Therefore, without loss of generality, we may assume $X = \operatorname{Spec} A$. Since f is quasi-compact and affine schemes are quasi-compact, as before X' is quasi-compact, and we can cover X' by a finite number of affine open subschemes $X'_\alpha = \operatorname{Spec} B_\alpha$ of X'. As before, let $\bar{X}'$ be the affine scheme $\bar{X}' := \coprod_\alpha X'_\alpha$. Let π, $\bar{f}$, p_i, $\bar{p}_i$, $i = 1, 2$, q, and $\bar{q}$ be as in the proof of the full faithfulness of F above.

We have canonical maps

$$\bar{X}' \times_{X'} \bar{X}' \xrightarrow{\delta} \bar{X}' \times_X \bar{X}' \xrightarrow{\pi\times\pi} X' \times_X X'.$$

If $(\mathscr{G},\, \varphi) \in \text{q-coh}_{(X' \to X)}$, then one checks that $(\pi^*\mathscr{G},\, (\pi \times \pi)^*\varphi)$ is a descent datum for the map $\bar{f} \colon \bar{X}' \to X$. Moreover, using the fact that φ restricted to the diagonal $X' \hookrightarrow X' \times_X X'$ is the identity map on $\mathscr{G}$, one checks that

$$\delta^*(\pi \times \pi)^*\varphi = \varphi^{\pi}_{\mathscr{G}}. \tag{1.2.1.1}$$

Since $\bar{f}$ is a map of affine schemes, by Theorem 2.1.1[1], the quasi-coherent sheaf $\pi^*\mathscr{G}$ descends to $\mathscr{F} \in \text{q-coh}_X$, and we can identify $(\bar{f}^*\mathscr{F},\, \varphi^{\bar{f}}_{\mathscr{F}})$ with the descent datum $(\pi^*\mathscr{G},\, (\pi \times \pi)^*\varphi)$. Applying δ^* to this identification, we get an identification of descent data with respect to the map $\pi \colon \bar{X}' \to X'$. In greater detail, the descent datum $\delta^*\varphi^{\bar{f}}_{\mathscr{F}}$ on $\bar{f}^*\mathscr{F} (= \pi^* f^* \mathscr{F})$ identifies with $\delta^*(\pi \times \pi)^*\varphi$ on $\pi^*\mathscr{G}$. Now, $\delta^*\varphi^{\bar{f}}_{\mathscr{F}} =$

[1]To be proved in the next lecture

$\varphi^{\pi}_{f^*\mathscr{F}}$. Using this and (3.2.1.1) we obtain an isomorphism $f^*\mathscr{F} \xrightarrow{\sim} \mathscr{G}$ by the full faithfulness of the functor $\mathscr{K} \mapsto (\pi^*\mathscr{K}, \varphi^{\pi}_{\mathscr{K}})$ on q-coh$_{X'}$ as proven in the discussion at the beginning of Subsection 3.2.[2] It remains to identify $\varphi^{f}_{\mathscr{F}}$ with φ under the just deduced isomorphism $f^*\mathscr{F} \xrightarrow{\sim} \mathscr{G}$. According Remark **??** this can be checked after applying $(\pi \times \pi)^*$ since $\pi \times \pi$ is faithfully flat. Doing this yields the original identification $(\bar{f}^*\mathscr{F}, \varphi^{\bar{f}}_{\mathscr{F}}) \xrightarrow{\sim} (\pi^*\mathscr{G}, (\pi \times \pi)^*\varphi)$, whence $(f^*\mathscr{F}, \varphi^{f}_{\mathscr{F}}) \xrightarrow{\sim} (\mathscr{G}, \varphi)$. It is clear that the process $(\mathscr{G}, \varphi) \mapsto \mathscr{F}$ is functorial in $(\mathscr{G}, \varphi) \in$ q-coh$_{(X' \to X)}$ and by its "construction" gives a pseudo-inverse to the functor F. □

Thus the affine case of Theorem 3.2.1 namely Theorem 2.1.1 implies the general case, and we are done.

References

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[2]Note that π is quasi-compact and faithfully flat.