LECTURE 3

1. From the last lecture

1.1. Recall the **Č**ech complex $C^{\bullet}_{B/A}(M)$ for an A-module M:

(1.1.1)
$$0 \to M \xrightarrow{\alpha_M} B \otimes_A M \xrightarrow{d^0} B^{\otimes 2} \otimes_A M \xrightarrow{d^1} \dots$$
$$\dots \xrightarrow{d^{r-2}} B^{\otimes r} \otimes_A M \xrightarrow{d^{r-1}} B^{\otimes r+1} \otimes_A M \xrightarrow{d^r} \dots$$

where $d^r = \sum_i (-1)^i e_i$ and

$$e_i(b_0 \otimes \dots \otimes b_r \otimes m) = b_o \otimes \dots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \dots b_r \otimes m.$$

For consistency we set $d^{-1} = \alpha_M$ and $B^{\otimes 0} = A$. The usual arguments give

$$d^r \circ d^{r-1} = 0, \qquad r \geq 0$$

We showed that this complex is exact. Moreover, we note that $d^0 \colon B \otimes_A M \to B^{\otimes 2} \otimes_A M$ is given by $b \otimes m \mapsto 1 \otimes b \otimes m - b \otimes 1 \otimes m$. Indeed, by definition, $d^0 = e_0 - e_1$ where $e_0(b \otimes m) = 1 \otimes b \otimes m$ and $e_1(b \otimes m) = b \otimes 1 \otimes m$. It follows that

(1.1.2)
$$M = \ker (e_0 - e_1)$$

2. Proof of faithful flat descent for affine schemes

2.1. The reader is now expected to look up the definition of a descent datum for a faithfully flat map of rings $A \to B$ from the previous lecture. We will now prove:

Theorem 2.1.1. *Suppose B is faithfuly flat over A. Then the functor $F \colon \mathrm{Mod}_A \to \mathrm{Mod}_{A \to B}$ defined above is an equivalence of categories.*

Theorem 2.1.1 asserts that for a B-module N to be of the form $B \otimes_A M$ for some A-module M, it is necessary and sufficent for N to carry a descent datum $\psi \colon N \otimes_A B \xrightarrow{\sim} B \otimes_A N$. In this case the module $M \in \mathrm{Mod}_A$ is unique up to isomorphism. In fact, as we will see later,

$$M = \{ n \in N \mid 1 \otimes n = \psi(n \otimes 1) \}.$$

The proof of *loc.cit.* is not difficult, being essentially a familiar Čech cohomology argument, suitably modified to the faithfully flat situation.

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2.2. Now suppose (N, ψ) is a descent datum on B/A. Let α and β be the maps

$$\alpha\colon N \to B \otimes_A N, \qquad n \mapsto 1 \otimes n;$$
$$\beta\colon N \to B \otimes_A N, \qquad n \mapsto \psi(n \otimes 1).$$

Define

$$M := \ker(\alpha - \beta). \tag{2.2.1}$$

We claim that there is an isomorphism

$$\theta(= \theta_{N,\psi})\colon (B \otimes_A M,\, \psi_M) \xrightarrow{\sim} (N,\, \psi)$$

in $\mathrm{Mod}_{A\to B}$, where $\psi_M\colon (B\otimes_A M)\otimes_A B \to B \otimes_A (B \otimes_A M)$ is the map given by $b \otimes m \otimes b' \mapsto b \otimes b' \otimes m$ (cf. Proposition II.2.2.3). Note that the claim implies, in particular, that $N \cong B \otimes_A M$.

Let $\theta\colon B \otimes_A M \to N$ be $b \otimes m \mapsto bf(m)$, where

$$f\colon M \hookrightarrow N$$

is the natural inclusion map of A-modules. It is clear that θ is functorial in (N, ψ) (since α and β are). We leave it to the reader to check that θ is a map of descent data, i.e., to check that the diagram

$$\begin{array}{ccc}
B \otimes_A M \otimes_A B & \xrightarrow{\theta\otimes 1} & N \otimes_A B \\
{\scriptstyle \psi_M}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
B \otimes_A B \otimes_A M & \xrightarrow[1\otimes\theta]{} & B \otimes_A N
\end{array}$$

commutes using the fact that by definition of M, $1 \otimes f(m) = \psi(f(m) \otimes 1)$.

Next, as in Picard II, let $\iota_M\colon M \otimes_A B \to B \otimes_A M$ be the natural map given by $m \otimes b \mapsto b \otimes m$. Consider the diagram with exact rows

$$\text{(D)}\qquad \begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_A B & \longrightarrow & N \otimes_A B & \xrightarrow{(\alpha-\beta)\otimes 1} & (B \otimes_A N) \otimes_A B \\
 & & {\scriptstyle \theta\circ\iota_M}\big\downarrow & & {\scriptstyle \psi}\big\downarrow & & \big\downarrow{\scriptstyle \psi_1} \\
0 & \longrightarrow & N & \xrightarrow[d^{-1}=\alpha_N]{} & B \otimes_A N & \xrightarrow[d^0=e_0-e_1]{} & B \otimes_A B \otimes_A N
\end{array}$$

where $\psi_1(b \otimes n \otimes b') = b \otimes \psi(n \otimes b')$ (cf. II.(2.2.1)). The rows of (D) are exact for the following reasons. First, by definition of M, $0 \to M \to N \xrightarrow{\alpha-\beta} B \otimes_A N$ is exact, and tensoring this with the *flat* A-algebra B gives us the top row of (D). The exactness of the bottom row of (D) follows from the exactness of $C^{\bullet}_{B/A}(N)$.

We claim that (D) commutes. As, before, it is convenient to denote the M in N by $f\colon M \hookrightarrow N$. We leave the commutatvity of the rectangle on the left to the reader. The following two facts are helpful for this. First, the image of $m \otimes b \in M \otimes_A B$ in $B \otimes_A N$ under the "south followed by east" route is $1 \otimes b(f(m)) \in B \otimes_A N$. To see this is also the image under the "east followed by south" route, use the fact that ψ is a $B^{\otimes 2}$-module map, whence $\psi((1 \otimes b)x) = (1 \otimes b)\psi(x)$.

The commutativity of the rectangle on the right uses the co-cycle rule namely

$$\psi_1 \circ \psi_3 = \psi_2 \tag{2.2.2}$$

which is the requirement for ψ to be a descent datum on N. Recall from II.(2.2.1) that $\psi_3(n \otimes b_1 \otimes b_2) = \psi(n \otimes b_1) \otimes b_2$ and $\psi_2(n \otimes b' \otimes b) = \sum_\alpha b^*_\alpha \otimes b' \otimes n^*_\alpha$ where

$\sum_\alpha b^*_\alpha \otimes n^*_\alpha = \psi(n \otimes b)$. In particular (with $b' = 1$ in the above formula for ψ_2) we have

$$\begin{aligned} \psi_2(n \otimes 1 \otimes b) &= \sum_\alpha b^*_\alpha \otimes 1 \otimes n^*_\alpha \\ &= e_1(\sum_\alpha b^*_\alpha \otimes n^*_\alpha) \\ &= (e_1 \circ \psi)(n \otimes b). \end{aligned} \tag{2.2.3}$$

We will show that

$$\psi_1 \circ (\alpha \otimes 1) = e_0 \circ \psi \tag{i}$$

and

$$\psi_1 \circ (\beta \otimes 1) = e_1 \circ \psi. \tag{ii}$$

The relation (i) is easy since $\alpha(n) = 1 \otimes n$ and $e_0(b \otimes n) = 1 \otimes b \otimes n$. We leave the details to the reader. The relation (ii) is trickier. Here are the details for (ii).

$$\begin{aligned} \psi_1 \circ (\beta \otimes 1)(n \otimes b) &= \psi_1(\beta(n) \otimes b) \\ &= \psi_1(\psi(n \otimes 1) \otimes b) \\ &= \psi_1(\psi_3(n \otimes 1 \otimes b)) \\ &= \psi_1 \circ \psi_3(n \otimes 1 \otimes b) \\ &= \psi_2(n \otimes 1 \otimes b) && \text{(by (2.2.2))} \\ &= e_1 \circ \psi(n \otimes b) && \text{(by (2.2.3))} \end{aligned}$$

In view of (i) and (ii) we get $\psi_1 \circ ((\alpha - \beta) \otimes 1) = (e_0 - e_1) \circ \psi$. Thus the rectangle on the right in diagram (D) commutes. Since the rows of (D) are exact and ψ and $\psi_1$are isomorphisms, $\theta \circ \iota_M$, whence θ, is also an isomorphism.

Clearly the assignment $(N, \psi) \mapsto M$ is functorial in $(N, \psi) \in \mathrm{Mod}_{A\to B}$. Moreover, it is evident from the above discussion, as well as (1.1.2) and (2.2.1), that it provides a pseudo-inverse to the functor $M \mapsto (B \otimes_A M, \psi_{_M})$ on Mod_A. This completes the proof of Theorem 2.1.1. □

Remark 2.2.4. Compare the above with the proof of gluing of sheaves.

References

[FGA] A. Grothendieck, *Fondements de la Géométrie Algébrique*, Sém, Bourbaki, exp. no° 149 (1956/57), 182 (1958/59), 190 (1959/60), 195(1959/60), 212 (1960/61), 221 (1960/61), 232 (1961/62), 236 (1961/62), Benjamin, New York, (1966).

[EGA] ——— and J. Dieudonné, *Élements de géométrie algébrique I*, Grundlehren Vol **166**, Springer, New York (1971).

[FGA-ICTP] B. Fantechi, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure, A. Vistoli, *Fundamental Algebraic Geometry, Grothendieck's FGA explained*, Math. Surveys and Monographs, Vol **123**, AMS (2005).

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergebnisse Vol **21**, Springer-Verlag, New York, 1980.

[M] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies **89**.