LECTURE 24

1. Sections of E/H and reductions of structure group to H

1.1. **Reduction of structure group and sections of E/H.** As before, let $H \hookrightarrow G$ be a closed subgroup scheme of G which *smooth* over S, and let $\pi\colon E \to X$ be a G-torsor, where $X \in \mathbb{S}\mathrm{ch}_{/S}$. For $T \in \mathbb{S}\mathrm{ch}_{/X}$, as usual $E_T := T \times_X E$ and $\pi_T\colon E_T \to T$ is the G-torsor over T obtained by base-change. In the last lecture we proved the following Theorem:

Theorem 1.1.1. *Let $T \in \mathbb{S}ch_{/X}$. There is a bijective correspondence between isomorphism classes of reductions of the structure group of $\pi_T\colon E_T \to T$ to H and X-maps $\varepsilon\colon T \to E/H$.*

Let us quickly recall why a section of E/H over $T \in \mathbb{S}\mathrm{ch}_{/X}$ gives us a reduction of structure group. So suppose $\varepsilon\colon T \to E/H$ is a map over X. According to *loc.cit.* ε is represented by a pair $(T' \xrightarrow{p} T,\, e)$ with p an fppf-map in $\mathbb{S}\mathrm{ch}_{/X}$ and $e\colon T' \to E$ a morphism in $\mathbb{S}\mathrm{ch}_{/X}$ such that there is an element $h \in H(T'')$ satisfying $e(p_2) = e(p_1)h$. The map e is equivalent to giving a trivialisation θ of $E_{T'} \to T'$ and the equation $e(p_2) = e(p_1)h$ is equivalent to saying that the resulting transition function (for the fppf-cover $p\colon T' \to T$) g_θ is equal to h. In other words we have a reduction of structure group of the G-torsor $\pi_T\colon E_T \to T$ to H. This reduction is independent of $(T \to T, e)$, i.e., the transition function from different representatives of ε are cohomologous as 1-cocycles taking values in $H(T'')$. This means that the element $h \in H(T'')$ gives us a well-defined H-torsor $r_\varepsilon\colon P_\varepsilon \to T$ which is independent of the data $(T' \to T, e)$. The map $H_{T'} \to E_{T'}$ given by $(t',\, h) \mapsto e(t')h$ is H-equivariant and a closed immersion. Hence it descends to give us a well defined closed immersion $P_\varepsilon \hookrightarrow E_T$ which is H-equivariant and hence a composite $P_\varepsilon \to E_T \to E$ which we denote q. Note that the H-equivariance of $P_\varepsilon \hookrightarrow E_T$ implies that the diagram

$$\begin{array}{ccc} P_\varepsilon & \xrightarrow{\;q\;} & E \\ {\scriptstyle r_\varepsilon}\downarrow & & \downarrow{\scriptstyle \varpi} \\ T & \xrightarrow[\;\varepsilon\;]{} & E/H \end{array}$$

Fix the data $(T' \to T, e)$ giving the map $\varepsilon T \to E/H$ above. Now if $Z \in \mathbb{S}\mathrm{ch}_{/X}$ and we have maps $t\colon Z \to T$ and $e_Z\colon Z \to E$ such that

$$\begin{array}{ccc} Z & \xrightarrow{\;e_Z\;} & E \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \varpi} \\ T & \xrightarrow[\;\varepsilon\;]{} & E/H \end{array}$$

commutes, then we have a unique map $\varphi\colon Z \to P_\varepsilon$ such that $r_\varepsilon \circ \varphi = t$ and $q \circ \varphi = e_Z$. Indeed if we set $Z' = Z \times_T T'$, then the natural map $Z' \to E_{T'}$, namely the

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composite $Z' \to T' \xrightarrow{(1_{T'},\, e)} E_{T'}$, takes values in $H_{T'}$ with respect to our inclusion $H_{T'} \hookrightarrow E_{T'}$. This descends give a map $Z \to P_\varepsilon$ with the required properties.

We have shown

Proposition 1.1.2. *With $\varepsilon\colon T \to E/H$ as above, $T \times_{E/H} E$ is representable by P_ε. More precisely, we have a cartesian diagram*

$$\begin{array}{ccc} P_\varepsilon & \xrightarrow{q} & E \\ {\scriptstyle r_\varepsilon}\downarrow & \square & \downarrow{\scriptstyle \varpi} \\ T & \xrightarrow[\varepsilon]{} & E/H \end{array}$$

2. **Closing Remarks**

2.1. E/H **is an algebraic space.** We saw in the last proposition that the base change of $\varpi\colon E \to E/H$ by a scheme $T \in \mathbb{S}\mathrm{ch}_{/X}$ is representable by a H-torsor over T. This amounts to saying that $E \to E/H$ is a smooth map. Moreover, we have seen that E/H can be regarded as the quotient of E by a smooth equivalence relation. A famous theorem of Michael Artin then says that E/H is an *algebraic space*, i.e., it is the quotient of the form Z/R, where $Z \in \mathbb{S}\mathrm{ch}_{/X}$ and R is an *étale equivalence relation* (see [A, Cor. (6.3), p. 184]).

2.2. **Zariksi triviality.** A famous theorem of Serre says that if $E \to X$ is a $GL_{n,S}$-torsor, then it is Zariski locally trivial. If S is the spectrum of an algebraically closed field and X is a smooth curve, then for arbitrary connected G (smooth and affine over k), then again a G-torsor is necessarily Zariski locally trivial.

2.3. If H is a smooth closed subgroup scheme of G, then it is possible that a trivial torsor has a non-trivial reduction of structure group to H. In other words it is possible that under the natural map $\mathrm{H}^1(X\, H_X) \to \mathrm{H}^1(X,\, G_X)$, non-trivial elements get mapped to the trivial element.

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