LECTURE 23

1. Comments on and solutions to selected problems in the midterm

1.1. **Problem 5.** *Suppose S is a scheme. Let $\mathfrak{M}$ be a collection of maps in $\mathbb{S}ch_{/S}$ which are stable under compositions, base changes, and which contains all isomorphisms. Let $F\colon (\mathbb{S}ch_{/S})^{\circ} \to$ **Sets** be a **Sets**-valued presheaf on $\mathbb{S}ch_{/S}$ such that $F(\emptyset) = \bigstar$, where $\bigstar$ our designated terminal object in* **Sets***. Define a topology $\tau_{\mathfrak{M}}$ on $\mathbb{S}ch_{/S}$ by decreeing that $\{U_i \to U\}$ is a cover if and only if the maps in the collection are jointly in $\mathfrak{M}$. Show that F is an $\mathfrak{M}$-sheaf if and only if it is a sheaf for the $\tau_{\mathfrak{M}}$ topology on $\mathbb{S}ch_{/S}$.*

Comment: Most of you missed the importance of the condition $F(\emptyset) = \bigstar$. It is used in showing that if F is a sheaf in the $\tau_{\mathfrak{M}}$-topology, then it is a *prepared presheaf*, i.e., $F(\coprod_\alpha T_\alpha) = \prod_\alpha F(T_\alpha)$. Let $T = \coprod_\alpha T_\alpha$ and let $p_\alpha\colon T_\alpha \to T$ be the canonical open immersion. One has a natural map $F(T) \to \prod_\alpha F(T_\alpha)$ given by $s \mapsto (p_\alpha^* s)$. We have to find a map $\prod_\alpha F(T_\alpha) \to F(T)$ which is the inverse of the map just described. First note that $\{T_\alpha \xrightarrow{p_\alpha} T\}$ is jointly in $\mathfrak{M}$ since the identity map $T \to T$ is in $\mathfrak{M}$. This means $\{T_\alpha \xrightarrow{p_\alpha} T\}$ is a cover for the $\tau_{\mathfrak{M}}$-topology. Let $(s_\alpha) \in \prod_\alpha F(T_\alpha)$ and let $\bigstar = \{\dagger\}$. It is clear that for all indices α and β, $T_{\alpha\beta} = T_\alpha \cap T_\beta = \emptyset$, whence $F(T_{\alpha\beta}) = \bigstar$. Thus $s_\alpha|_{T_{\alpha\beta}} = \dagger = s_\beta|_{T_{\alpha\beta}}$. Since F is a $\tau_{\mathfrak{M}}$-sheaf, this gives rise to a unique $s \in F(T)$ such that $p_\alpha^* s = s_\alpha$ for every index α.

1.2. **Problem 6.** *Let $f\colon X \to Y$ be a map of schemes. Show that the following are equivalent:*

(a) *Every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.*

(b) *There exists a covering (in the classical sense of the term) $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.*

(c) *The map f is surjective and given a point $x \in X$, there exists an open neighborhood U of x in X such that $f(U)$ us open in Y, and the map $U \to f(U)$ induced by f (by restricting to U) is a quasi-compact map.*

(d) *The map f is surjective and given a point $x \in X$, there exists a quasi-compact open neighborhood U of x in X such that $f(U)$ is open and affine in Y.*

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Brief Solution:

(a)⇒(b) is obvious.

(b)⇒(c). Suppose (b) is true. Then it is clear that f is surjective. Let $x \in X$ be a point. Let $y = f(x)$. Since (b) is true, there is an affine open neighbourhood V of y such that V is the image under f of a quasi compact open set U' in X. Pick an affine open neighborhood U'' of x contained in $f^{-1}(V)$. Set $U = U' \cup U''$. It is evident that $f(U) = V$. Moreover U is the finite union of quasi compact schemes, and hence is a quasi-compact scheme. Since U is quasi-compact and $V = f(U)$ is affine, the induced map $U \to f(U) = V$ is quasi-compact. Thus f satisfies (c).

(c)⇒(d). Suppose f satisfies (c). The map f is surjective. Let $x \in X$. We can find an open neighborhood U' of x such that $V' = f(U')$ is open and the resulting map $U' \to V'$ is quasi-compact. Call this map g. Let $y = f(x)$. Then $y \in V'$. Pick an affine open neighbourhood V of x such that $V \subset V'$. Set $U = g^{-1}(V)$. Since g is quasi-compact, and V is quasi-compact (being affine), U is also quasi-compact. Moreover $f(U) = V$. Thus f satisfies (d).

(d)⇒(a). First we make the following observation. Suppose f satisfies (d). Then there are open covers $\mathscr{U}' = \{U'_\alpha\}$ of X and $\mathscr{V}' = \{V'_\alpha\}$ of Y such that $f(U'_\alpha) = V'_\alpha$ and the map $g_\alpha\colon U'_\alpha \to V'_\alpha$ obtained by restricting f to U'_α, is quasi-compact for every index α. Indeed, we may take the indices α to be the points of X is necessary. (In slightly greater detail, for every point $x \in X$, we can find a quasi-compact open neighborhood U'_x of x such that $V'_x := f(U'_x)$ is open and affine in Y and clearly $U'_x \to V'_x$ is quasi-compact. The surjectivity of f ensures that the V'_x cover Y.)

Now suppose V is a quasi-compact open subscheme of Y. Set $V_\alpha = V'_\alpha \cap V$, and $U_\alpha = g_\alpha^{-1}(U_\alpha)$. Then $U_\alpha \to V_\alpha$ is quasi-compact since g_α is so. Denote this map by $f_\alpha\colon U_\alpha \to V_\alpha$. For each index α we have an affine open cover $\{V_{\alpha j}\}_j$ of V_α. For indices α and j in our range, set $U_{\alpha j} := f_\alpha^{-1}(U_{\alpha j})$. Since f_α is quasi-compact, each $U_{\alpha j}$ is quasi-compact. Now $\{V_{\alpha j}\}_{\alpha, j}$ is an affine open cover of the quasi-compact scheme V. It follows we have a finite number of affine open subsets $V_{\alpha_k j_k}$, $k = 1, \ldots, p$ which cover V. Set $U = \cup_{k=1}^p U_{\alpha_k j_k}$. Being a finite union of quasi-compact open sets, U is quasi-compact and open. It is clear that $f(U) = V$. Thus f satisfies (a).

1.3. **Problem 8.** *Let S be a scheme and let*

$$F\colon \left(\mathbb{S}ch_{/S}\right)^{\circ} \to \mathbf{Sets}$$

be a presheaf on $\mathbb{S}ch_{/S}$ such that $F(\emptyset) = \bigstar$. Let $\mathfrak{M}'$ be the class of faithfully flat and quasi-compact *maps, $\mathfrak{M}''$ the class of maps $V \to U$ such that $V = \coprod_i U_i$ with each U_i an open subscheme of U and $V \to U$ the natural map (open immersion on each U_i) and such that $U = \cup_i U_i$. Let $\mathfrak{M}$ be the class of fpqc-maps. Show that the following are equivalent:*

(a) *F is an $\mathfrak{M}'$-sheaf and an $\mathfrak{M}''$-sheaf.*

(b) *F is an $\mathfrak{M}$-sheaf.*

Brief Solution: It is clear (b)⇒(a), for maps in $\mathfrak{M}'$ and $\mathfrak{M}''$ are in $\mathfrak{M}$. Let us prove (a)⇒(b). Suppose F is an $\mathfrak{M}'$-sheaf as well as an $\mathfrak{M}''$-sheaf. Suppose $p\colon T' \to T$ is in $\mathfrak{M}$. According to Problem 6 (especially part (d)), p is faithfully flat and one can find a (Zariski) open cover $\{U_\alpha\}$ of T' by quasi-compact open subschemes, and a (Zariski) open cover $\{V_\alpha\}$ of T by affine open subschemes with

$p(U_\alpha) = V_\alpha$. Let $f_\alpha \colon U_\alpha \to V_\alpha$ be the map obtained by restricting p to U_α. Then f_α is in $\mathfrak{M}'$ since it is faithfully flat and quasi-compact. Now suppose $\xi \in F(T')$ is such that $p_1^*(\xi) = p_2^*(\xi)$. Let $\xi_\alpha = \xi|_{U_\alpha}$. Since $p_1^*(\xi) = p_2^*(\xi)$, the "restriction" of ξ_α to $U_\alpha \times_{V_\alpha} U_\alpha$ via either projection is the same. Since $f_\alpha \colon U_\alpha \to V_\alpha$ is in $\mathfrak{M}'$ and F is an $\mathfrak{M}'$-sheaf, it follows that we have a unique element $\zeta_\alpha \in F(V_\alpha)$ such that $f_\alpha^*(\zeta_\alpha) = \xi_\alpha$. These considerations hold for every index α. Next fix two indices α and β and write $V = V_\alpha \cap V_\beta$ and $U = f_\alpha^{-1}(V) \cup f_\beta^{-1}(V)$. Let $f \colon U \to V$ be the resulting map (i.e., f is obtained by restricting $p \colon T' \to T$ to U). Let $\xi_U = \xi|_U$. Then the equation $p_1^*(\xi) = p_2^*(\xi)$ implies that the pull-back of ξ_U to $U \times_V U$ by either projection is the same. Now $f \colon U \to V$ is quasi-compact since f_α and f_β are. Hence we have $\zeta_U \in F(V)$ such that $f^*(\zeta_U) = \xi_U$. Now $\xi_U|_{f_\alpha^{-1}(V)} = \xi_\alpha|_{f_\alpha^{-1}(V)}$ and hence by uniqueness of $\zeta_\alpha|_V$ we have $\zeta_U = \zeta_\alpha|_V$. Similarly $\zeta_U = \zeta_\beta|_V$. Thus $\zeta_\alpha|_{V_{\alpha\beta}} = \zeta_\beta|_{V_{\alpha\beta}}$ Since F is an $\mathfrak{M}''$-sheaf, the ζ_α glue uniquely to give $\zeta \in F(T)$. Moreover $p^*\zeta|_{U_\alpha} = \xi_\alpha$ for every α. Again using the fact that F is an $\mathfrak{M}''$-sheaf, we see that $p^*\zeta = \xi$.

1.4. **Problem 9 (b).** *Let $Z' \hookrightarrow T'$ be a closed subscheme of T' such that $p_1^{-1}(Z') = p_2^{-1}(Z')$ as closed subschemes of T''. Show that there is a unique closed subscheme $Z \hookrightarrow T$ such that $p^{-1}(Z) = Z'$.*
Comment: One has to use the fact that closed subschemes of T are the same as surjective maps of $\mathscr{O}_T$-algebras of the form

$$\mathscr{O}_T \twoheadrightarrow \mathscr{A}.$$

Moreover, if $(\mathscr{A}_1', \phi_1) \to (\mathscr{A}_2', \phi_2)$ is a map of descent data on T' with the underlying map of quasi-coherent sheaves $\mathscr{A}_1' \to \mathscr{A}_2'$ a map of $\mathscr{O}_{T'}$-algebras, then the resulting map of descended $\mathscr{O}_T$-modules, $\mathscr{A}_1 \to \mathscr{A}_2$ is a map of $\mathscr{O}_T$-algebras, as can be readily verified from the proof of faithfully flat descent. From part (a) of the problem, it is clear that the natural surjection $\mathscr{O}_{T'} \twoheadrightarrow \mathscr{O}_{Z'}$ is a map of descent data, and if $\mathscr{A}$ is the quasi-coherent sheaf sheaf such that $p^*\mathscr{A} = \mathscr{O}_{Z'}$ then $\mathscr{A}$ is an $\mathscr{O}_T$-algebra and the descended map $\mathscr{O}_T \to \mathscr{A}$ is surjective. This defines a unique closed subscheme Z of T such that $\mathscr{A} = \mathscr{O}_Z$. It is evident that $p^{-1}(Z) = Z'$ since $p^*(\mathscr{A}) = \mathscr{O}_{Z'}$.

1.5. **Problem 10.** *Let S be a scheme. In $\mathbb{S}ch_{/S}$ consider the cartesian diagram*

$$\begin{array}{ccc} T'' & \xrightarrow{p_2} & T' \\ {\scriptstyle p_1}\downarrow & \square & \downarrow{\scriptstyle p} \\ T' & \xrightarrow[p]{} & T \end{array}$$

with $p \colon T' \to T$ fpqc.

(a) *Let $f' \colon T' \to Z$ be a map in $\mathbb{S}ch_{/S}$ such that $f' \circ p_1 = f' \circ p_2$. Show that there exists a unique map of schemes $f \colon T \to Z$ such that $f' = f \circ p$. [*Hint*: Use Problem 8 to reduce to the case where p is faithfully flat and quasi-compact. Next reduce to the case where T and T' and Z are affine. Finally use the graph $\Gamma' = \Gamma_{f'} \hookrightarrow T' \times_S Z$ (show it is a closed subscheme of the product scheme!) and make it "descend" to a closed subschema of $T \times_S Z$. And then?]*

(b) *Conclude that $h_Z = \mathrm{Hom}_{\mathbb{S}ch_{/S}}(-, Z)$ is an fpqc-sheaf on $\mathbb{S}ch_{/S}$.*

Comments: The reduction to the affine case in part (a) was not done very well by any of you. First note that (a) and (b) are really equivalent statements. One is saying the same thing. According to Problem 8 we only have to check that h_Z is an $\mathfrak{M}'$-sheaf, since it is trivially an $\mathfrak{M}''$-sheaf (we are using the notations of the statement of Problem 8). In other words we have to show part (a) under the assumption that $p\colon T' \to T$ is faithfully flat and quasi-compact. By Proposition 2.1.1 of Lecture 6, we know that a set $C \subset T$ is open if and only if $p^{-1}(C)$ is open in T'. We will show that the problem is local on Z (surprising!). First observe that the condition $f'(p_1) = f'(p_2)$ implies that for any $z \in Z$, ${f'}^{-1}(z)$ is the union of fibres of p. (This amounts to the statement that if $f'(x'_1) = z$ and $x'_2 \in X'$ is a point such that $p(x'_1) = p(x'_2)$, then $f'(x'_2) = z$.) In particular, if $W \subset Z$, then ${f'}^{-1}(W) = p^{-1}(C)$ for $C = p({f'}^{-1}(W))$. In view of the just cited Proposition, if W is open in Z, then C must be open in X. One can therefore replace X by C, X' by ${f'}^{-1}(W)$ and Z by W in this case, and a solution to the problem in this situation amounts to a solution in the general situation. Thus without loss of generality, we may assume Z is affine. The problem is clearly local on X. So next we may assume X is affine. Since p is quasi-compact, X' is quasi-compact and so can be covered by a finite number of affine open subschemes. Replacing X' be the disjoint union of these finite number of affines if necessary, we may assume X' is affine.

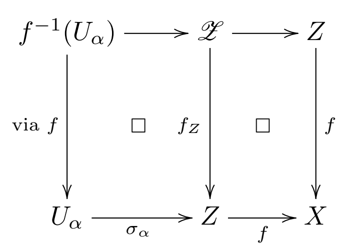
Finally if $A \to B$ is a map of rings, the the graph of the corresponding map of affine schemes is the closed subscheme of $\operatorname{Spec} A \otimes_{\mathbb{Z}} B$ given by the surjection $A \otimes_{\mathbb{Z}} B \to B$, $a \otimes b \mapsto ab$. Thus in this situation the graph is closed.

1.6. **Problem 14 (d).** *Let $f\colon Z \to X$ be a G-space over X. Consider $\mathscr{Z} := Z_Z = Z \times_X Z$, and the induced map $f_Z\colon \mathscr{Z} \to Z$. Define*

$$\Psi\colon G_Z \to \mathscr{Z}$$

by $(z, g) \mapsto (z, zg)$, $z \in Z$, $g \in G$. Suppose $f\colon Z \to X$ has local sections, *i.e., around each point $x \in X$ there is an open neighborhood such that the restriction $f^{-1}U_x \to U_x$ of f has a section. Suppose further that Ψ is an isomorphism. Show that $f\colon Z \to X$ is a principal bundle and the right G-action on Z induced by its principal bundle structure is the given right G-action on it.*

Comments: Let $\{\sigma_\alpha\colon U_\alpha \to f^{-1}(U_\alpha)\}_\alpha$ be the local sections of $f\colon Z \to X$. Note that $\{U_\alpha\}$ is an open cover of X. Now the base change of any trivial principal G-bundle is again t rival. Since Ψ is an isomorphism, $f_Z\colon \mathscr{Z} \to Z$ is a trivial principal G-bundle. Its base change via the map $\sigma_\alpha\colon U_\alpha \to Z$ is clearly $f^{-1}(U_\alpha)$. Thus $f^{-1}(U_\alpha) \to U_\alpha$ is a trivial principal G-bundle. By Problem 12, $f\colon Z \to X$ is a principal G-bundle.



2. Sections of E/H and reductions of structure group of $E \to X$

As before, let $H \hookrightarrow G$ be a closed subgroup scheme over S such that $H \to S$ is smooth.

Let $X \in \mathbb{S}\mathrm{ch}_{/S}$ and let $\pi\colon E \to X$ be a G-torsor. As in earlier lectures, E/H is defined to be fppf-sheafifcation of the presheaf $T \mapsto E(T)/H(T)$ on $\mathbb{S}\mathrm{ch}_{/X}$. We then have maps $\varpi\colon E \to E/H$ and $\pi_{/H}\colon E/H \to X$ such that $\pi = \pi_{/H} \circ \varpi$.

We discuss very briefly the strategy for showing that sections of $\pi_{/H}\colon E/H \to X$ correspond to reductions of structure group of $\pi\colon E \to X$. More generally, we will argue maps $T \to E$ of X-spaces give reductions of structure group of $\pi_T\colon E_T \to T$ to H. We will flesh this out in subsequent lectures.

Recall from Yoneda that a map of X-spaces $T \to E/H$, with $T \in \mathbb{S}\mathrm{ch}_{/X}$, is the same as a section of the sheaf E/H over T. Recall Proposition 1.3.2 of Lecture 21, namely: *Elements of $(E/H)(T)$ are represented by pairs $(T' \xrightarrow{p} T,\, e)$, where $e \in E(T')$ is an element such that $e(p_2) = e(p_1)h$ for a T''-valued point h of H. Two such pairs $(T_1,\, e_1)$ and $(T_2,\, e_2)$ represent the same element of $(E/H)(T)$ if and only if $e_2(t_2) = e_1(t_1)h$ where $t_i\colon T_1 \times_T T_2 \to T_i$, $i = 1, 2$, are the projections, and $h \in H(T_1 \times_T T_2)$.*

If $(T' \xrightarrow{p} T,\, e)$ is such a pair representing an element $\xi \in (E/H)(T)$ and $h \in H(T'')$ the element defined by $e(p_2) = e(p_1)h$, then h is a H-valued 1-cocycle. In fact if $t'' = (t_1',\, x_2') \in T''(W)$ for $W \in \mathbb{S}\mathrm{ch}_{/T}$, then the equation $e(p_2) = e(p_1)h$ translates to $e(t_2') = e(t_1')h(t_1',\, t_2')$, from which one can easily deduce that for three W-valued points t_1', t_2', and t_3' of T' lying over $W \to T$, the identity $h(t_1',\, t_2')h(t_2',\, t_3') = h(t_1',\, t_3')$ holds.

If $\widetilde{e} := (1_{T'},\, e)$, then $\widetilde{e}\colon T' \to E_{T'}$ is a section of the G-torsor $\pi_{T'}\colon E_{T'} \to T'$, and gives a trivialization of this torsor. The corresponding transition function is precisely h. Thus we have reduced the structure group of E_T to H by means of the pair $(T' \xrightarrow{p} T,\, e)$. It is not hard to see that if we have another pair representing $\xi\colon T \to E/H$, then that pair will give rise to a transition function, i.e., a 1-cocycle, $h^* \in H(T'')$ which is cohomologous (via an element in $H(T')$) to h. For this we again use Proposition 1.3.2 of Lecture 21. The conclusion is that the element of $\mathrm{H}^1(T,\, H_T)$ obtained from any pair $(T' \to T,\, e)$ representing $\xi \in (E/H)(T)$ is independent of the pair. In other words $\xi\colon T \to E/H$ gives us a well-defined reduction of structure group of $\pi_T\colon E_T \to T$.

Let the $P_\xi \to T$ be the H-torsor corresponding to the element in $\mathrm{H}^1(T,\, H_T)$ obtained from $\xi\colon T \to E/H$. We will show in subsequent lectures that $P_\xi = T \times_{E/H} E$.

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