LECTURE 22

1. E/H and $E(G/H)$

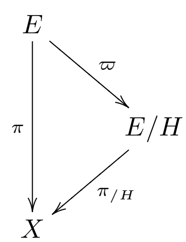
Throughout this section $H \hookrightarrow G$ is a closed subgroup scheme (over S) of the groups scheme $G \to S$, such that $H \to S$ is smooth.

1.1. **Recap from the last lecture.** Recall that in Lecture 21 we defined E/H as an fppf-sheaf together with natural maps $\varpi\colon E \to E/H$ and $\pi_{/H}\colon E/H \to X$. In greater detail, E/H is the fppf-sheafication of the presheaf $F\colon (\mathbb{S}\mathrm{ch}_{/X})^{\circ} \to (\mathrm{Sets})$ on $\mathbb{S}\mathrm{ch}_{/X}$ given by

$$T \mapsto E(T)/H(T)$$

as T varies in $\mathbb{S}\mathrm{ch}_{/X}$. Note that we are working with objects and maps in $\mathbb{S}\mathrm{ch}_{/X}$, and hence $X(T)$ is a singleton set for every $T \in \mathbb{S}\mathrm{ch}_{/X}$. (In particular $X(E) = \{\pi\}$.) One therefore has natural maps of functors $E \to F$ and $F \to X$ such that the composite $E \to F \to X$ is $\pi\colon E \to X$. Sheafifying with respect to the fppf-topology on $\mathbb{S}\mathrm{ch}_{/X}$ and using the fact that E and X are already sheaves, we get a commutative diagram

(1.1.1)



Setting $X = S$ and $E = G$ we recover the construction of G/H given in Subsection 2.1 of Lecture 20, with $\varpi = p_H$ and $\pi_{/H} = t_H$.

1.2. **Representability of E/H.** As we saw E/H is not always representable. Indeed G/H is often not representable. In Lecture 21 we gave certain sufficient conditions for G/H to exist as a scheme (see Examples 1.2.1 of loc.cit.). If further, G/H is a locally quasi-affine G-space, then E/H is representable. We state the result formally below.

Proposition 1.2.1. *Suppose G/H is representable as a locally quasi-affine G-space. Then E/H is representable by the X-scheme $E(G/H)$.*

Proof. The statement is obvious if $X = S$ and $E = G$. Note that for $T \in \mathbb{S}\mathrm{ch}_{/X}$, $E_T/H = T \times_S (E/H) = E_T/H_T$. In particular for $T \in \mathbb{S}\mathrm{ch}_{/S}$, we have the relations $G_T/H = T \times_S (G/H) = G_T/H_T$. Let $p\colon X' \to X$ be a trivialising fpqc-covering map for the G-torsor $\pi\colon E \to X$, with (say) the trivialisation

$$\theta\colon X' \times_S G = G_{X'} \xrightarrow{\sim} X' \times_X E =: E'.$$

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As before, let $g_\theta\colon X'' \to G$ denote the transition function. Let $p_1, p_2\colon X'' \rightrightarrows X'$ be the two projections. From our just made observations we have an isomorphism

$$\theta_{/H}\colon X' \times_S (G/H) \xrightarrow{\ \sim\ } E'/H$$

such that the automorphism $p_1^*(\theta_{/H})^{-1} \circ p_2^*(\theta_{/H})$ of the X''-scheme $X'' \times_S (G/H)$ is given by $(x'',\, \xi) \mapsto (x'',\, g_\theta\xi)$. In other words $E/H = E(G/H)$. $\square$

1.3. **The space $E \times_S H$ and the sheaf $E \times_{E/H} E$.** Let $T \in \mathbb{S}\mathrm{ch}_{/X}$ and let $e\colon T \to E$ be a T-valued point of the X-scheme E. If $h \in H(T)$, then it is evident that e and eh have the same image in $(E/H)(T)$ (since they have the same image in $E(T)/H(T)$). Therefore $(e,\, eh) \in (E \times_{E/H} E)(T)$. One can therefore define a map of sets $(E \times_S H)(T) \to (E \times_{E/H} E)(T)$ by $(e,\, h) \mapsto (e,\, eh)$. This is clearly functorial and gives us a map of fppf-sheaves

$$E \times_S H \to E \times_{E/H} E. \tag{1.3.1}$$

Proposition 1.3.2. *The map $E \times_S H \to E \times_{E/H} E$ of* (1.3.1) *above is an isomorphism.*

Proof. Consider the map $\beta\colon E \times_S H \to E \times_X E$ given by $(e,\, h) \mapsto (e,\, eh)$. It is evident that for each $T \in \mathbb{S}\mathrm{ch}_{/X}$, this defines an injective map $(E \times_S H)(T) \to (E \times_S E)(T)$. For each $T \in \mathbb{S}\mathrm{ch}_{/X}$, define the subset $R(T)$ of $(E \times_X E)(T)$ by

$$R(T) = \{(e,\, eh) \in (E \times_X E)(T) \mid h \in H(T)\}.$$

Clearly R is a presheaf on $\mathbb{S}\mathrm{ch}_{/X}$. Two facts are immediate. First, the map $\beta(T)$ gives us a bijection $(E \times_S H)(T) \xrightarrow{\ \sim\ } R(T)$ which is functorial in $T \in \mathbb{S}\mathrm{ch}_{/X}$, i.e., R is representable and $E \times_S H \xrightarrow{\ \sim\ } R$. Second, from its definition $R(T)$ is an equivalence relation on $E(T)$ for every $T \in \mathbb{S}\mathrm{ch}_{/X}$, whence R is a scheme theoretic equivalence relation. Now E/R is defined as the fppf-sheafification of $T \mapsto E(T)/R(T)$. In other words $E/R = E/H$ by definition, for $E(T)/R(T) = E(T)/H(T)$. At this point, recall the Proposition 1.3.2 from Lecture 21 characterizing sections of E/H over an X-scheme T, namely: *The elements of $(E/H)(T)$ are represented by pairs $(T' \xrightarrow{p} T,\, e)$, where $e \in E(T')$ is an element such that $e(p_2) = e(p_1)h$ for a T''-valued point h of H. Two such pairs $(T_1,\, e_1)$ and $(T_2,\, e_2)$ represent the same element of $(E/H)(T)$ if and only if $e_2(t_2) = e_1(t_1)h$ where $t_i\colon T_1 \times_T T_2 \to T_i$, $i = 1, 2$, are the projections, and $h \in H(T_1 \times_T T_2)$.* Now suppose $(e_1,\, e_2) \in (E \times_{E/H} E)(T)$, for some $T \in \mathbb{S}\mathrm{ch}_{/X}$. In other words e_1 and e_2 are two T valued points of E such that their images in $(E/H)(T)$ are the same. Taking $T_1 = T_2 = T$ in Proposition 1.3.2 of Lecture 21, we see that there exists a unique element $h \in H(T)$ such that $e_2 = e_1 \cdot h$. Thus the valued point $(e_1, e_2) = (e_1,\, e_1 \cdot h)$ and hence is an element of $R(T)$. Conversely, it is obvious that if $h \in H(T)$, then $(e,\, eh)$ is in $(E \times_{E/H} E)(T)$ for all $e \in E(T)$. Thus $R = E \times_{E/H} E$. But we just saw that $E \times_S H \xrightarrow{\ \sim\ } R$. This completes the proof. $\square$

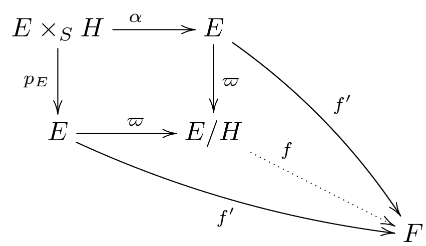
Remark 1.3.3. Let $q_1, q_2\colon R \rightrightarrows E$ be the two projections. Proposition 1.3.2 says in particular that we have a commutative diagram with each rectangle cartesian:

$$\begin{array}{ccccccc}
E \times_S H & \xrightarrow{\ \sim\ } & E \times_{E/H} E & = & R & \xrightarrow{\ q_2\ } & E \\
\downarrow & & & & \downarrow{\scriptstyle q_1} & \square & \downarrow{\scriptstyle \varpi} \\
E & & = & & E & \xrightarrow[\ \varpi\ ]{} & E/H
\end{array}$$

Next we will show that E/H is a quotient in the category of fppf-sheaves on $\mathbb{S}\mathrm{ch}_{/X}$, i.e., it has the co-equalizer property for the maps $R \rightrightarrows E$. We state it somewhat differently as follows.

Proposition 1.3.4. *Let $\alpha\colon E \times_S H \to E$ be the action map of H on E, and $p_E\colon E \times_S H \to E$ the first projection. In other words $\alpha(e, h) = eh$ and $p_E(e, h) = e$. Suppose $F\colon (\mathbb{S}ch_{/X})^\circ \to (\mathrm{Sets})$ is an fppf-sheaf on $\mathbb{S}ch_{/X}$ and $f\colon E \to F$ a map of sheaves such that $f' \circ \alpha = f \circ p_E$. Then there exists a unique map $f\colon E/H \to F$ such that $f \circ \varpi = f'$. In other words in the following diagram in which the solid arrows form a commutative diagram, the dotted arrow can be filled uniquely to make the whole diagram commutative.*

(1.3.4.1)



We postpone the proof. It depends on the following facts (some of which you will be asked to prove in your homework):

- The map $\varpi\colon E \to E/H$ is relatively representable, i.e., if $T \in \mathbb{S}\mathrm{ch}_{/X}$ and $T \to E/H$ is an X-map, then $T \times_{E/H} E$ is representable.
- If $T \to E/H$ is as above, the map $T \times_{E/H} E \to T$ is an H-torsor.
- Therefore the map $\varpi\colon E \to E/H$ is a surjective map of fppf-sheaves. A map $f\colon A \to B$ of $\mathfrak{M}$-sheaves (where $\mathfrak{M}$ is a collection of maps in $\mathbb{S}\mathrm{ch}_{/S}$ which contains all isomorphisms, is stable under compositions and base changes) is said to *surjective* if given $\xi \in B(T)$, there is a map $p\colon T' \to T$ in $\mathfrak{M}$ and an element $\eta \in A(T')$ such that $p^*\xi = f(T)(\eta)$.
- If $A \to B$ is a surjective map of $\mathfrak{M}$-sheaves (see above), then B is the co-equalizer of $A \times_B A \rightrightarrows A$.

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