LECTURE 21

1. Sections of $E(F)$ and G-equivariant maps

Let X be an S-scheme, $\pi\colon E\to X$ a G-torsor, and $F\to S$ a locally quasi-affine G-scheme. As usual $\pi_F\colon E(F)\to X$ will denote the associated fibre bundle over X.

1.1. **Equivariant maps.**

Recall that in Lecture 18, Theorem 1.1.3, we gave an alternative description of $E(F)$ as the quotient by G of $E\times_S F$ under the right action

$$(e,\, f)\cdot g=(eg,\, g^{-1}f)$$

on $E\times_S F$, where as usual, $e\colon T\to E$, $g\colon T\to G$, and $f\colon T\to F$ are T-valued points of E, G, and F, for $T\in \mathbb{S}\mathrm{ch}_{/S}$. If $F=S$, then easy considerations show that $E(F)=X$ (we point out that any S-automorphism of S is the identity, and so any G-action on S must be trivial). Indeed if $(p\colon X'\to X,\, \theta\colon G_{X'}\xrightarrow{\sim}E_{X'})$ is a trivializing data for $\pi\colon E\to X$, and g_θ the corresponding transition element, then g_θ acts trivially on $X''\times_S F=X''\times_S S=X''$, and the descent of the X'-scheme $X'\times_S F=X'\times_S S=X'$ to X via the descent datum provided by g_θ is necessarily X itself. Thus $X=E(S)=E\times_S S/G=E/G$. We record the two formulae

$$E(F)=(E\times_S F)/G \text{ and } E/G=X. \tag{1.1.1}$$

Moreover, the quotient $E(F)=(E\times_S F)/G$ is an effective quotient of $E\times_S F$ by a smooth (and hence fpqc) equivalence relation. This last condition means that if $R\to(E\times_S F)\times_S(E\times_S F)$ is the scheme theoretic equivalence relation $(e,\, f)\sim(eg,\, g^{-1}f)$, then the two projections $R\rightrightarrows E\times_S F$ are smooth [1]. The universal property of quotients then implies that if $q\colon E\times_S F\to(E\times_S F)/G=E(F)$ is the quotient map and if $\varphi\colon E\times_S F\to Z$ is a G-equivariant map in $\mathbb{S}\mathrm{ch}_{/S}$ for the trivial action of G on Z, then there is a unique map of $\phi\colon E(F)\to Z$ such that $\varphi=\phi\circ q$.

Proposition 1.1.2. *There is a bijective correspondence between sections of the fibre-bundle $\pi_F\colon E(F)\to X$ and G-equivariant maps $E\to F$. Here the action on F is the right action on it induced by the given left action on it (i.e., $f\cdot g:=g^{-1}f$).*

Proof. Suppose $\varphi\colon E\to F$ is G-equivariant in $\mathbb{S}\mathrm{ch}_{/S}$, and suppose as before

$$q\colon E\times_S F\to(E\times_S F)/G=E(F)$$

is the quotient map. We have a map $\widetilde{\varphi}\colon E\to E\times_S F$ given by $\widetilde{\varphi}=(1_E,\,\varphi)$. Note that $\widetilde{\varphi}$ is a section of the projection $E\times_S F\to E$. Clearly $\widetilde{\varphi}$ is G-equivariant. Hence, so is $q\circ\widetilde{\varphi}\colon E\to E(F)$. Since the action of G on $E(F)$ is trivial, by the universal property of the quotient $\pi\colon E\to E/G=X$, we deduce a map from $\psi\colon X=E/G\to E(F)$ such that $\phi\circ\pi=q\circ\widetilde{\varphi}$, and such a ϕ is unique. Since $\widetilde{\varphi}$ is

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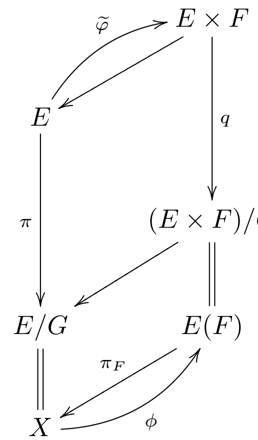
Date: October 24, 2012.

[1] R can be identified with $E\times_X E\times_S F$ and the projections q_1 and q_2 to $E\times_S F$ by the assignments $(e_1,\, e_2,\, f)\mapsto(e_1,\, g_u(e_1,\, e_2)f)$ and $(e_1,\, e_2,\, f)\mapsto(e_2,\, f)$ respectively. Here the map $g_u\colon E\times_X E\to G$ is the transition function corresponding to the trivialisation of $E\to X$ by the fpqc-covering $E\to X$ and the diagonal section of $p_1\colon E\times_X E\to E$.

a section of $E \times_S F \to E$, one checks easily that ϕ is a section of $\pi_F \colon E(F) \to X$. As in Lecture 18, Subsection 2.1, the situation is summarized by the following commutative diagram



In summary, given a G-equivariant map $\varphi \colon E \to F$, we obtain a section $\phi \colon X \to E(F)$ of π_F.

Note that the parallelograms in the above diagram are cartesian. Therefore if $\phi \colon X \to E(F)$ is a section of π_F, then we have a map $\phi \circ \pi \colon E \to E(F)$, whence a map $\widetilde{\varphi} \colon E \to E \times_X E(F)$ given by $e \mapsto (e, \phi \circ \pi(e))$. Identifying $E \times_X E(F)$ with $E \times_S F$, we see that $\widetilde{\varphi}$ is a G-equivariant section of $E \times_S F \to E$, and hence the map $\varphi = p_F \circ \widetilde{\varphi}$ is G-equivariant, where $p_F \colon E \times_S F \to F$ is the projection on to F. Thus the section $\phi \colon X \to E(F)$ gives rise to a G-equivariant map $\varphi \colon E \to F$. It is easy to see that the two processes are inverses.

To help the reader verify details, we point out that the identification

$$E \times_X E(F) = E \times_S F$$

used above is given by $(e, [e', f]) \mapsto (e, gf)$ where g is the unique valued point of G such that $e' = eg$. $\square$

1.2. **The space E/H.** Next suppose $H \subset G$ is a closed subgroup scheme over S such that $H \to S$ is smooth. We wish to say that reductions of structure group of the G-torsor $\pi \colon E \to X$ are in one-to-one correspondence with sections of $E/H = E(G/H) \to X$.

The Problem: Does G/H exist as a scheme? How about E/H? We did define G/H earlier as an fppf sheaf.

Examples 1.2.1. Regarding the question of representability of quotients by schemes, here are some examples:

(1) If S is the spectrum of a field k, then G/H exists as a *smooth* quasi-projective variety. If $k \to K$ is a field extension with K algebraically closed, then $G/H(K) = G(K)/H(K)$. Here is the idea of the proof. One has to find a representation $G \to GL(V)$, for a finite dimensional k-vector-space V, such that H is the stabilizer of a line L in V. We therefore have an action of G on $\mathbb{P}(V)$. If $x_0 \in \mathbb{P}(V)$ is the point represented by L, then the stabilizer of x_0 is H. One shows that G/H can be realised as the G-orbit of x_0 with its reduced structure and that this reduced orbit is smooth and locally closed in $\mathbb{P}(V)$. Details can be found in [C1, p. 45, Thm. 18.1.1].

(2) In the above case, if H in the above situation is a normal subgroup then G/H is affine [C1, p. 46, Example 18.1.4].
(3) Suppose $G \to S$ is a reductive group scheme and $H \to S$ is a *parabolic* subgroup of G, i.e. $P_{\overline{s}}$ is a parabolic subgroup of $G_{\overline{s}}$, for every $s \in S$ and $\overline{s}$ a geometric point over s. Then it turns out G/P exists as a smooth projective S-scheme [C2, p. 128, Cor. 5.2.8]. Moreover the G-space G/P can be covered by open subschemes which are G-stable and affine over S. Thus if $\pi\colon E \to S$ is a G-torsor, $E/P = E(G/P)$ exists as smooth proper scheme over X.

Just as we defined G/H as an fppf-sheaf, we can define E/H as an fppf-sheaf together with natural maps $\varpi\colon E \to E/H$ and $\pi_{/H}\colon E/H \to X$. In greater detail, consider the pre sheaf

$$F\colon (\mathbb{S}\mathrm{ch}_{/X})^{\circ} \to (\mathrm{Sets})$$

given by

$$T \mapsto E(T)/H(T)$$

as T varies in $\mathbb{S}\mathrm{ch}_{/X}$. Set

$$E/H = F^{+}$$

where F^{+} is the sheafification of F with respect to the fppf-topology on $\mathbb{S}\mathrm{ch}_{/X}$. Note that we are working with objects and maps in $\mathbb{S}\mathrm{ch}_{/X}$, and hence $X(T)$ is a singleton set for every $T \in \mathbb{S}\mathrm{ch}_{/X}$. (In particular $X(E) = \{\pi\}$.) One therefore has natural maps of functors $E \to F$ and $F \to X$ such that the composite $E \to F \to X$ is $\pi\colon E \to X$. Sheafifying with respect to the fppf-topology on $\mathbb{S}\mathrm{ch}_{/X}$ and using the fact that E and X are already sheaves, we get a commutative diagram

(1.2.2)
$$\begin{array}{ccc} E & & \\ & \searrow^{\varpi} & \\ \downarrow{\scriptstyle \pi} & & E/H \\ & \swarrow_{\pi_{/H}} & \\ X & & \end{array}$$

1.3. **Sections of** E/H**.** To understand sections of E/H we need the following Lemma.

Lemma 1.3.1. *Suppose $T \in \mathbb{S}ch_{/S}$, $g \in G(T)$, and we have an fpqc-map $t\colon T' \to T$ such that $g(t) \in H(T')$. Then $g \in H(T)$.*

Proof. It is clear we may assume S is affine (whence so are G and H) by replacing S by an affine open subscheme (such subschemes cover S) and taking inverse images of this open subscheme in all the schemes we are considering (SG, S, T, T'). Further, without loss of generality, we may assume $T = \operatorname{Spec} A$. We know that there is a quasi-compact open subscheme U of T' such that $t(U) = T$. Replacing T' by U if necessary, we may assume T' is quasi-compact. Finally, replacing T' by a finite disjoint union of affine open subschemas if necessary, we may assume $T' = \operatorname{Spec} B$. The graph of $g(t)\colon T' \to G$ is a closed subscheme of $T' \times_S G$. Call this graph Γ'. Let the graph of $g\colon T \to G$ be $\Gamma \hookrightarrow T \times_S G$. Clearly $\Gamma' = t_G^{-1}(\Gamma)$, where $t_G\colon T' \times_S G \to T \times_S G$ is the base change of $t\colon T' \to T$. By our hypothesis, Γ' is actually a closed subscheme of $T' \times_S H$, i.e., we have a hierarchy of inclusions

$\Gamma' \hookrightarrow T' \times_S H \hookrightarrow T' \times_S G$. We therefore have a hierarchy of surjective maps of $\mathscr{O}_{T'\times_S G}$-modules given by $\mathscr{O}_{T'\times_S G} \twoheadrightarrow \mathscr{O}_{T'\times_S H} \twoheadrightarrow \mathscr{O}_{\Gamma'}$. Since the first arrow and the composite arrow are maps of descent data, and since the arrows are surjective, the second arrow is also a map of descent data. It follows that we have a sequence of surjective $\mathscr{O}_{T\times_S G}$-maps $\mathscr{O}_{T\times_S G} \twoheadrightarrow \mathscr{O}_{T\times_S H} \twoheadrightarrow \mathscr{O}_{\Gamma}$. (These maps are surjective because t_G is faithfully flat.) The last surjection says $\Gamma \hookrightarrow T\times_S H$, which is another way of saying $g \in H(T)$. □

By the sheafification process described in Section 3 of Lecture 7, a section of the sheaf E/H over $T \in \mathbb{S}\mathrm{ch}_{/X}$ is represented by a pair $(T' \xrightarrow{p} T, [e])$ where $p\colon T' \to T$ is an fppf-map in $\mathbb{S}\mathrm{ch}_{/X}$, e is a T'-valued point of E, and $[e] = eH(T')$ is the image of e in $E(T')/H(T')$. This pair has the property that there is an fppf-map $q\colon \widetilde{T} \to T''$ such that $q^*p_1^*[e] = q^*p_2^*[e]$. Let $g \in G(T'')$ be the unique element such that $e(p_2) = e(p_1)g$. It follows that $e(p_2\circ q) = e(p_1\circ q)g(q)$. On the other hand, the relation $q^*p_1^*[e] = q^*p_2^*[e]$ is equivalent to saying that $e(p_2\circ q) = e(p_1\circ q)h$ for some (necessarily unique) element $h \in H(\widetilde{T})$. It follows that $g(q) = h$. From the Lemma we conclude that $g \in H(T'')$.

Moreover, by modifying the above argument slightly, pairs $(T_1 \to T, [e_1])$ and $(T_2 \to T, [e_2])$ represent the same element of $(E/H)(T)$, precisely when the equation $e_2(t_2) = e_1(t_1)h$ for some (necessarily unique) $h \in H(T_1 \times_T T_2)$, where $t_i\colon T_1 \times_T T_2 \to T_i$ $(i = 1, 2)$ are the two projections.

We have proven the following:

Proposition 1.3.2. *Let $T \in \mathbb{S}ch_{/X}$. Then elements of $(E/H)(T)$ are represented by pairs $(T' \xrightarrow{p} T, e)$, where $e \in E(T')$ is an element such that $e(p_2) = e(p_1)h$ for a T''-valued point h of H. Two such pairs (T_1, e_1) and (T_2, e_2) represent the same element of $(E/H)(T)$ if and only if $e_2(t_2) = e_1(t_1)h$ where $t_i\colon T_1 \times_T T_2 \to T_i$, $i = 1, 2$, are the projections, and $h \in H(T_1 \times_T T_2)$.*

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