LECTURE 20

1. **Trivialisations again**

Let $X \in \mathbb{S}\mathrm{ch}_{/S}$ and let $\pi\colon E \to X$ be a G-torsor. Recall Lemma 2.1.4 from Lecture 14, namely: *Let e_1, e_2 two sections of π. Then there exists a unique element $g_{12} \in G(X)$ such that*

$$e_2 = e_1 g_{12}.$$

Moreover, if $\psi_{12}\colon G_X \xrightarrow{\sim} G_X$ is the G-equivariant isomorphism of X-schemes [1] *given by $\psi_{12} = \psi_{e_1}^{-1} \circ \psi_{e_2}$ then ψ_{12} is described by $(x, g) \mapsto (x, g_{12}(x)g)$ for valued points x of X and g of G having the same source.*

We can make the same statement about sections of principal bundles. So suppose G is a topological group. As usual assume, for the rest of this section, that all topological spaces, including G, are Hausdorff, and all group actions are continuous. Sections of principal bundles are assumed continuous unless otherwise stated. Suppose $\pi\colon E \to X$ is a principal G-bundle. We know that $E \to X$ is trivial if and only if it has a section $\sigma\colon X \to E$ and trivialisations $\theta\colon G_X \xrightarrow{\sim} E$ are in bijective correspondence with such sections. Indeed given a trivialization θ, the canonical identity section of $G_X \to X$ translates to a section of $E \to X$, and conversely a section $\sigma\colon X \to E$ gives rise to the isomorphism $\psi_\sigma\colon G_X \xrightarrow{\sim} E$ given by $(x, g) \mapsto \sigma(x)g$. Let $\theta_i\colon G_X \xrightarrow{\sim} E$, $i = 1, 2$, be two isomorphisms of principal G-bundles, and $\sigma_i\colon X \to E$ the corresponding sections. Let $g_{12}\colon X \to G$ be the continuous map such that

$$\sigma_2(x) = \sigma_1(x)g_{12}(x),$$

and let $\psi_{12} G_X \xrightarrow{\sim} G_X$ be the automorphism of principal G bundles given by $\psi_{12} = \theta_1^{-1} \circ \theta_1$. Then g_{12} can also be characterized as the continuous map such that ψ_{12} is $(x, g) \mapsto (x, g_{12}(x)g)$. In other words, if $\theta_1(x, g_1) = \theta_2(x, g_2)$ then

$$g_1 = g_{12}(x)g_2,$$

whereas,

$$\sigma_2(x) = \sigma_1(x)g_{12}(x).$$

Note that $\theta_1(x, g_1) = \theta_2(x, g_2)$ is equivalent to $\psi_{12}(x, g_2) = (x, g_1)$.

2. **Reduction of structure group for G-torsors**

Now suppose $X \in \mathbb{S}\mathrm{ch}_{/S}$ and $\pi\colon E \to X$ is a G-torsor. Let $H \subset G$ be a closed subscheme of G smooth over S. We say the structure group of E is reducible to H if there exists an fpqc-map $X' \to X$ and a trivialization $\theta\colon G_{X'} \xrightarrow{\sim} E_{X'}$ such that the transition function

$$g_\theta\colon X'' \to G$$

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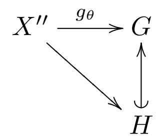
[1] With respect to the right G-action on G_X.

is factors through H:



This is equivalent to saying that for each $T \in \mathbb{S}\mathrm{ch}_{/S}$, $g_\theta(T)\colon X''(T) \to G(T)$ takes values in the subgroup $H(T)$ of $G(T)$.

Suppose $\mathrm{H}^1(X, G_X)$ and $\mathrm{H}^!(X, H_X)$ are the first cohomology sets for the group schemes G and H respectively, and

$$h_*\colon \mathrm{H}^1(X, H_X) \to \mathrm{H}^1(X, G_X)$$

the natural map. Let $\pi\colon E \to X$ be a G-torsor and $\zeta \in \mathrm{H}^1(X, G_X)$ be the element corresponding to $E \to X$. Then the structure group of $\pi\colon E \to X$ is reducible to H if and only if ξ in the image of h_*, i.e., $\zeta = h_*\xi$ for some $\xi \in \mathrm{H}^1(X, H_X)$. A choice of such a ξ is a reduction of the structure group of the G-torsor $E \to X$ to H. Note, if $\xi \in \mathrm{H}^1(X, H_X)$ is a reduction of the structure group of E to H, then ξ gives rise to a H-torsor, $p\colon P \to X$ and a H-equivariant closed immersion $i\colon P \hookrightarrow E$ of X-schemes. The latter data nails ξ.

We wish to characterize reductions of structure group of E to H by sections of $E(G/H) \to X$ (or $E/H \to X$, since (hopefuly) $E(G/H) = E/H$ as in the classical sutuation).

Here are the possible problems one could encounter:

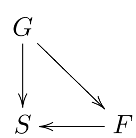
(1) G/H may not exist as an S-scheme.
(2) Even if it exists, G/H may not be a locally quasi-affine space over S
(3) E/H may not exist as a scheme.

2.1. **The space G/H.** First consider the presheaf $F\colon (\mathbb{S}\mathrm{ch}_{/S})^\circ \to \mathbf{Sets}$ on $\mathbb{S}\mathrm{ch}_{/S}$ given by

$$T \mapsto G(T)/H(T).$$

We point out that for any $T \in \mathbb{S}\mathrm{ch}_{/S}$, the set $S(T)$ is a single element set, and hence we have map of presheaves $F \to S$. We also have a map of presheaves $G \to F$ and the following diagram commutes:

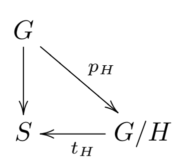
(2.1.1)



Definition 2.1.2. The quotient G/H is the fppf-sheafification of the pre sheaf F.

Note that the maps $G \to F$ and $F \to S$ in (2.1.1) gives rise to the maps $p_H\colon G \to G/H$ and $t_H\colon G/H \to S$ such that the diagram

(2.1.3)



commutes.

Suppose $\mathscr{F}$ is an fppf-sheaf on $\mathbb{S}\mathrm{ch}_{/S}$ such that G acts trivially on the right on $\mathscr{F}$ (i.e., $G(T)$ acts trivially on the right on $\mathscr{F}(T)$ for every $T \in \mathbb{S}\mathrm{ch}_{/S}$) and we have a G-equivariant map $\varphi\colon G \to \mathscr{F}$ for the right action of G on G. Then for each T we have a unique map $\psi(T)\colon G(T)/H(T) \to \mathscr{F}(T)$ such that the following diagram commutes

$$\begin{array}{ccc} G(T) & \xrightarrow{\varphi(T)} & \mathscr{F}(T) \\ & \searrow & \uparrow{\scriptstyle \psi(T)} \\ & & G(T)/H(T) \end{array} \tag{2.1.4}$$

It is evident that $\psi(T)$ is functorial in T as T varies over $\mathbb{S}\mathrm{ch}_{/S}$. By the universal property of sheafification (or by using the functoriality of sheafifications) we get a unique map

$$\varphi_{/H}\colon G/H \to \mathscr{F}$$

such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \mathscr{F} \\ & {\scriptstyle p_H}\searrow & \uparrow{\scriptstyle \varphi_{/H}} \\ & & G/H \end{array} \tag{2.1.5}$$

3. **The Yoneda Lemma**

The major point of this section is to interpret elements of $F(X)$ (for $X \in \mathbb{S}\mathrm{ch}_{/S}$ and F a contrvariant functor **Sets**-valued functor on $\mathbb{S}\mathrm{ch}_{/S}$) as maps

$$X \to F.$$

Here X in the above relation is identified with the functor h_X and the arrow $X \to F$ as a natural transformation of functors. Note that if $F = h_Y$ where $Y \in \mathbb{S}\mathrm{ch}_{/S}$, then such a correspondence is a tautology if we identify h_Y with Y.

3.1. Let $\widehat{\mathbb{S}\mathrm{ch}}_{/S}$ denote the category of contravariant **Sets**-valued functors on $\mathbb{S}\mathrm{ch}_{/S}$, i.e the objects are contravariant functors on $\mathbb{S}\mathrm{ch}_{/S}$ taking values in **Sets** and morphisms are natural transformations of such functors. Recall that the functor $X \to h_X$, for $X \in \mathbb{S}\mathrm{ch}_{/S}$ is a fully faithful embedding of $\mathbb{S}\mathrm{ch}_{/S}$ into $\widehat{\mathbb{S}\mathrm{ch}}_{/S}$.

Let $F \in \widehat{\mathbb{S}\mathrm{ch}}_{/S}$ (i.e., $F\colon (\mathbb{S}\mathrm{ch}_{/S})^\circ \to \mathbf{Sets}$ is a functor) and $X \in \mathbb{S}\mathrm{ch}_{/S}$. Note that $h_X(X) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(X, X)$ has a distinguished element, namely the identity map 1_X. Therefore a map

$$\varphi\colon h_X \to F$$

in $\widehat{\mathbb{S}\mathrm{ch}}_{/S}$ gives rise to an element $\xi_\varphi \in F(X)$, namely the image under $\varphi(X)$ of 1_X.

The process can be reversed. To see this let $\xi \in F(X)$. For a set A, we have a map

$$e_\xi(A)\colon \mathrm{Hom}_{\mathbf{Sets}}(F(X), A) \to A$$

given by "evaluation at ξ", i.e., the map $f \mapsto f(\xi)$ for $f\colon F(X) \to A$. Note that $e_\xi(A)$ is functorial in A, and gives a functorial map

$$e_\xi\colon \mathrm{Hom}_{\mathbf{Sets}}(F(X), -) \longrightarrow \mathbf{1}_{\mathbf{Sets}}$$

of endo-functors on **Sets**. We therefore have a sequence of natural transformations

$$h_X = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(-,\, X) \longrightarrow \mathrm{Hom}_{\mathbf{Sets}}(F(X),\, F(-)) \xrightarrow{e_\xi(F(-))} F.$$

Let

$$\widehat{\xi}\colon h_X \to F$$

be the above composite. It is easy to check that the assignments $\varphi \mapsto \xi_\varphi$ and $\xi \mapsto \widehat{\xi}$ are inverses of each other. Hence we get bijective correspondence of sets

$$\mathrm{Hom}_{\widehat{\mathbb{S}\mathrm{ch}}_{/S}}(h_X,\, F) \xrightarrow{\sim} F(X) \tag{3.1.1}$$

given by $\varphi \mapsto \xi_\varphi$. The correspondence (3.1.1) is called the *Yoneda correspondence*. We often do not distinguish between $\xi \in F(X)$ and the map $\widehat{\xi}\colon h_X \to F$, or between h_X and X. Hence $\xi \in F(X)$ is thought of as a map $\xi\colon X \to F$. If $Y \in \mathbb{S}\mathrm{ch}_{/S}$ and $F = h_Y$ (i.e., $F = Y$ in line with our identifications) then we are simply stating the obvious, or, more precisely, restating the fully faithful embedding $h_{(-)}\colon \mathbb{S}\mathrm{ch}_{/S} \to \widehat{\mathbb{S}\mathrm{ch}}_{/S}$ given by $Y \mapsto h_Y$.

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