LECTURE 2

All rings are commutative with a multiplicative identity, and all ring maps (i.e., ring homomorphisms) are unital (i.e., 1 maps to 1). If A is a ring, then by an A-algebra we always mean a commutative A-algebra.

1. Descent data for Modules

1.1. **Notations.** For any ring A the category of A-modules will be denoted Mod_A. Now suppose B is an A-algebra and M is an A-module. Then the map $M \otimes_A B \to B \otimes_A M$ of B-modules given by $m \otimes b \mapsto b \otimes m$ will be denoted ι_M. Note that we have:

$$\iota_M \colon M \otimes_B A \xrightarrow{\sim} B \otimes_A M.$$

With A and B as above, and $M \in \mathrm{Mod}_A$, set

(i) $B^{\otimes r} := \underbrace{B \otimes_A \cdots \otimes_A B}_{r \text{ times}}$.

(ii) $\alpha_M \colon M \to B \otimes_A M$, $m \mapsto 1 \otimes m$.

1.2. **Descent data.** Fix an A-algebra B as above. Every B-module N gives rise to two $B^{\otimes 2}$-modules, namely

(i) $N \otimes_A B$ with module structure $(b_1 \otimes b_2)(n \otimes b) = (b_1 n) \otimes (b_2 b)$;
(ii) $B \otimes_A N$ with module structure $(b_1 \otimes b_2)(b \otimes n) = (b_1 b) \otimes (b_2 n)$.

Similarly we have three $B^{\otimes 3}$-modules, namely $N \otimes_A B \otimes_A B$, $B \otimes_A N \otimes_A B$, and $B \otimes_A B \otimes_A N$, the $B^{\otimes 3}$-module structures being obvious and along the lines of the $B^{\otimes 2}$-module structures described above. Suppose we have a $B^{\otimes 2}$-map

$$\psi \colon N \otimes_A B \to B \otimes_A N.$$

We have three maps induced by ψ described as follows:

(1.2.1)

$$\psi_{23} \colon B \otimes_A N \otimes_A B \to B \otimes_A B \otimes_A N; \qquad \psi_{23} = \mathrm{id}_B \otimes \psi,$$
$$\psi_{13} \colon N \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A N; \qquad \psi_{13} = (\mathrm{id}_B \otimes \iota_N) \circ (\psi \otimes \mathrm{id}_B) \circ (\mathrm{id}_N \otimes \iota_B),$$
$$\psi_{12} \colon N \otimes_A B \otimes_A B \to B \otimes_A N \otimes_A B; \qquad \psi_{12} = \psi \otimes \mathrm{id}_B.$$

Note that if $\psi(n \otimes b) = \sum_\alpha b^*_\alpha \otimes n^*_\alpha$, then $\psi_{13}(n \otimes b_1 \otimes b) = \sum_\alpha b^*_\alpha \otimes b_1 \otimes n^*_\alpha$.

Definition 1.2.2. Let $N \in \mathrm{Mod}_B$. A *descent datum* on N is an isomorphism $\psi \colon N \otimes_A B \xrightarrow{\sim} B \otimes_A N$ such that with ψ_{12}, ψ_{13}, ψ_{23} as in (1.2.1), we have

$$\psi_{13} = \psi_{23} \circ \psi_{12}$$

as maps from $N \otimes_A B \otimes_A B$ to $B \otimes_A B \otimes_A N$. (This is the so-called *cocycle rule.*) *The category of B-modules with descent data (for A)* is the category $\mathrm{Mod}_{A \to B}$

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whose objects are pairs (N, ψ) with $N \in Mod_B$ and ψ a descent datum, and whose morphisms $(N, \psi) \xrightarrow{\beta} (N', \psi')$ are B-maps $\beta\colon N \to N'$ such that the diagram

$$\begin{array}{ccc} N \otimes_A B & \xrightarrow{\psi} & B \otimes_A N \\ {\scriptstyle \beta\otimes \mathrm{id}_B}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B \otimes \beta} \\ N' \otimes_A B & \xrightarrow[\psi']{} & B \otimes_A N' \end{array}$$

commutes.

Given an A-module M, there is a very natural descent datum on $B \otimes_A M$, namely the map

$$\psi_M\colon (B \otimes_A M) \otimes_A B \to B \otimes_A (B \otimes_A M)$$

given by $b \otimes m \otimes b' \mapsto b \otimes b' \otimes m$.

Proposition 1.2.3. $(B \otimes_A M, \psi_M) \in \mathrm{Mod}_{A\to B}$. *Moreover, if* $M \to M'$ *is an* A*-map then the induced map* $\beta\colon B \otimes_A M \to B \otimes_A M'$ *defines a map in* $\mathrm{Mod}_{A\to B}$.

This is an easy (and obvious) computation, which we leave to the reader. Thus the assignment $M \mapsto (B \otimes_A M, \psi_M)$ gives us a functor

$$F\colon \mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}.$$

The theorem of faithful flat descent for affine schemes, i.e. the theorem that follows, says that this assignment is an equivalence of categories.

Theorem 1.2.4. *Suppose* B *is faithfuly flat over* A*. Then the functor* $F\colon \mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}$ *defined above is an equivalence of categories.*

We will prove Theorem 1.2.4 in the next lecture. *Loc.cit.* asserts that for a B-module N to be of the form $B \otimes_A M$ for some A-module M, it is necessary and sufficent for N to carry a descent datum $\psi\colon N \otimes_A B \xrightarrow{\sim} B \otimes_A N$. In this case the module $M \in \mathrm{Mod}_A$ is unique up to isomorphism. In fact, as we will see later,

$$M = \{n \in N \mid 1 \otimes n = \psi(n \otimes 1)\}.$$

The proof of *loc.cit.* is not difficult, being essentially a familiar Čech cohomology argument, suitably modified to the faithfully flat situation.

2. The Čech complex for faithfully flat algebras

Throughout this section we fix a ring A, an A-module M, and a faithfully flat A-algebra B.

2.1. Define a sequence of A-maps

$$0 \to M \xrightarrow{\alpha_M} B \otimes_A M \xrightarrow{d^0} B^{\otimes 2} \otimes_A M \xrightarrow{d^1} \dots \qquad (2.1.1)$$

$$\dots \xrightarrow{d^{r-2}} B^{\otimes r} \otimes_A M \xrightarrow{d^{r-1}} B^{\otimes r+1} \otimes_A M \xrightarrow{d^r} \dots$$

where $d^r = \sum_i (-1)^i e_i$ and

$$e_i(b_0 \otimes \dots \otimes b_r \otimes m) = b_o \otimes \dots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \dots b_r \otimes m.$$

For consistency write $d^{-1} = \alpha_M$ and $B^{\otimes 0} = A$. The usual arguments give

$$d^r \circ d^{r-1} = 0, \qquad r \geq 0$$

whence (2.1.1) defines a complex of A-modules which we denote $C^\bullet_{B/A}(M)$.

Proposition 2.1.2. *$C^\bullet_{B/A}(M)$ is exact.*

Proof. Suppose we have a "retract" of the algebra structure map $\alpha_A\colon A \to B$, i.e. a map of rings $g\colon B \to A$ such that the composite $g \circ \alpha_A$ is the identity. (In other words, suppose $\operatorname{Spec} B \to \operatorname{Spec} A$ has a section.) For $r \geq -1$ define

$$k_r\colon B^{\otimes r+2} \otimes_A M \to B^{\otimes r+1} \otimes_A M$$

by

$$b_0 \otimes \cdots \otimes b_{r+1} \otimes m \mapsto g(b_0)b_1 \otimes \cdots \otimes b_{r+1} \otimes m.$$

Set $k_{-2} = 0$. One checks that

$$k_r d^r + d^{r-1} k_{r-1} = 1$$

for $r \geq -1$. Thus $\{k_r\}$ is a contracting homotopy on $C^\bullet_{B/A}(M)$, whence, in this case, the assertion follows.

For an A-algebra A', let $B' := B \otimes_A A'$. Then, as is easily checked, for $r \geq 1$ $B' \otimes_{A'} B' \otimes_{A'} \cdots \otimes_{A'} B' = B^{\otimes r} \otimes_A A'$, where the number of tensor factors on the left is r. In other words $B'^{\otimes r} = B^{\otimes r} \otimes_A A'$. This identity is easily seen to hold for $r = 0$ and $r = -1$ also. It is then obvious that

$$C^\bullet_{B/A}(M) \otimes_A A' = C^\bullet_{B'/A'}(M \otimes_A A') \qquad (*)$$

Now suppose A' is faithfully flat over A, and $C^\bullet_{B'/A'}(M \otimes_A A')$ is exact. Then by $(*)$ and faithful flatness, it follows that $C^\bullet_{B/A}(M)$ is also exact. Set $A' = B$. Then $B' = B^{\otimes 2}$, and the structure map $\alpha_{A'}\colon A' \to B'$ is $b \mapsto b \otimes 1$. Clearly the map $g'\colon B' \to A'$ given by $b_1 \otimes b_2 \mapsto b_1 b_2$ is a retract of $\alpha_{A'}$. Thus, as we saw earlier in this proof, $C^\bullet_{B'/A'}(M \otimes_A A')$ is exact. But A' is faithfully flat over A, since $A' = B$. Hence we are done. □

Remark 2.1.3. Note that $d^0\colon B \otimes_A M \to B^{\otimes 2} \otimes_A M$ is given by

$$b \otimes m \mapsto 1 \otimes b \otimes m - b \otimes 1 \otimes m.$$

Indeed, by definition, $d^0 = e_0 - e_1$ where $e_0(b \otimes m) = 1 \otimes b \otimes m$ and $e_1(b \otimes m) = b \otimes 1 \otimes m$. It follows that

$$M = \ker(e_0 - e_1) \qquad (2.1.3.1)$$