LECTURE 19

1. **Cohomologous 1-cocycles and isomorphisms of descent data**

Recall that if G is a (Hausdorff) topological group and X is a Hausdorff space, then two G-valued 1-cocycles $(g^{(1)}_{\alpha\beta})$ and $(g^{(2)}_{\alpha\beta})$ (for the same open cover $\mathscr{U} = \{U_\alpha\}$ of X) are cohomologous if, for each index α, there are continuous maps $t_\alpha \colon U_\alpha G$ such that for every $x \in U_{\alpha\beta}$ and every pair of indices (α, β) we have the relationship $g^{(2)}_{\alpha\beta}(x)t_\beta(x) = t_\alpha(x)g^{(1)}_{\alpha\beta}(x)$. In the schemes case, working with torsors for the group scheme $G \to S$, this is exactly analogous to the notion of a G-*equivariant* of descent data as we shall see.

1.1. **The topological case.** In this subsection G is a topological space, and all topological spaces occurring (including G) are Hausdorff, and all group actions are continuous.

Let X be a topological space. If

$$g \colon U \to G$$

is a continuous map from an open subset U of X, let

$$\widetilde{g} \colon U \times G \to U \times G$$

be the map

$$(u,\, g^*) \mapsto (u,\, g(u)g^*).$$

Note that $\widetilde{g}$ is an automorphism of the trivial principal G-bundle $G_U \to U$. Conversely, as is well known, every principal G-bundle automorphism of G_U must of if the form $\widetilde{g}$ for some continuous map $g \colon U \to G$. Indeed, if $\psi \colon G_U \xrightarrow{\sim} G_U$ is such an automorphism, and $\epsilon \in G$ the identity element, then for every $u \in U$ we have $g(u) \in G$ determined by second coordinate of $\psi(u, \epsilon)$. In other words $\psi(u, \epsilon) = (u, g(u))$. G-equivariant then forces the identity $\psi = \widetilde{g}$. Moreover such a g is unique for a given ψ as is clear from the above considerations.

Now suppose $\mathscr{U} = \{U_\alpha\}$ is an open cover of X. Let $(g^{(1)}_{\alpha\beta})$ and $(g^{(2)}_{\alpha\beta})$ be G-valued 1-cocycles for this cover. Recall that the give isomorphic principal G-bundles if and only if they are cohomologous. Now, they are cohomologous if for each index α, there are continuous maps $t_\alpha \colon U_\alpha G$ such that for every $x \in U_{\alpha\beta}$ and every pair of indices (α, β) we have the relationship

$$g^{(2)}_{\alpha\beta}(x)t_\beta(x) = t_\alpha(x)g^{(1)}_{\alpha\beta}(x). \tag{1.1.1}$$

Now we have various automorphism of principal G-bundles, namely

$$\widetilde{t}_\alpha \colon G_{U_\alpha} \xrightarrow{\sim} G_{U_\alpha},$$

and for $i = 1, 2$

$$\widetilde{g}^{(i)}_{\alpha\beta} \colon G_{\alpha\beta} \xrightarrow{\sim} G_{\alpha\beta}.$$

Date: October 19, 2012.

The condition (1.1.1) can be re-written as the commutativity of

(1.1.2)
$$\begin{array}{ccc} G_{\alpha\beta} & \xrightarrow[\sim]{\widetilde{g}^{(1)}_{\alpha\beta}} & G_{\alpha\beta} \\ {\scriptstyle \widetilde{t}_\beta}\downarrow\wr & & \wr\downarrow{\scriptstyle \widetilde{t}_\alpha} \\ G_{\alpha\beta} & \xrightarrow[\widetilde{g}^{(2)}_{\alpha\beta}]{\sim} & G_{\alpha\beta} \end{array}$$

Let $X' = \coprod_\alpha U_\alpha$ and set $\widetilde{t} = \coprod_\alpha \widetilde{t}_\alpha$. We have an automorphism of trivial principal G-bundles

$$\widetilde{t}\colon G_{X'} \xrightarrow{\sim} G_{X'}.$$

Each of the 1-cocycles $(g^{(i)}_{\alpha\beta})$ gives rise to a well-known "glueng" equivalence relation $\sim_i$ on X'. Diagram (1.1.2) says that $x'_1 \sim_1 x'_2$ if and only if $\widetilde{t}(x'_1) \sim_2 \widetilde{t}(x'_2)$. Thus the resulting quotients $X'/\sim_1$ and $X'/\sim_2$ are isomorphic principal G-bundles on X, something we've seen earlier.

If we write $X'' = X' \times_X X'$ and $p_i\colon X'' \to X$, $i = 1, 2$ for the two projections, then the cocylcle $(\widetilde{g}^{(i)}_{\alpha\beta})$ gives rise to an automorphism $\widetilde{g}^{(i)}\colon G_{X''} \xrightarrow{\sim} G_{X''}$, for $i = 1, 2$. The commutativity of Diagram (1.1.2) is equivalent to the commutativity of

(1.1.3)
$$\begin{array}{ccc} G_{X''} & \xrightarrow[\sim]{\widetilde{g}^{(1)}} & G_{X''} \\ {\scriptstyle p_2^*(\widetilde{t})}\downarrow\wr & & \wr\downarrow{\scriptstyle p_1^*(\widetilde{t})} \\ G_{X''} & \xrightarrow[\widetilde{g}^{(2)}]{\sim} & G_{X''} \end{array}$$

This is exactly analogous to an isomorphism of (equivariant) descent data, namely the descent data given by the two co-cycles under discussion.

1.2. **The case of schemes.** Now consider our standard situation of a smooth relatively affine group scheme $G \to S$. If $U \in \mathbb{S}\mathrm{ch}_{/S}$ and $g \in G(U)$ then we have a an automorphism of trivial G-torsors

$$\widetilde{g}\colon G_U \to G_U$$

given by $(u,\, g^*) \mapsto (u\, g(u) g^*)$, where u is a T-valued point of U and g^* a T-valued point of G.

Conversely any automorphism of the G-torsor G_T must be of the form $\widetilde{g}$ for a unique $g \in G(T)$, as in the topological situation (the proof is exactly analogous to the proof in that situation).

Let $p\colon X' \to X$ be an fpqc-map in $\mathbb{S}\mathrm{ch}_{/S}$. By what we have just seen, an automorphism of X''-schemes $\psi\colon G_{X''} \to G_{X''}$ is G-equivariant (i.e., it is an automorphism of trivial G-torsors) if and only if

$$\psi = \widetilde{g}$$

for a unique element $g \in G(X'')$. Moreover, ψ is a descent datum if and only if for every valued point (x'_1, x'_2, x'_3) of X''', the co-cycle rule $g(x'_1,\, x'_2) g(x'_2, x'_3) = g(x'_1, x'_3)$ holds. In other words g is a 1-cocycle or a transition element and gives rise to a G-torsor on X.

Now suppose we have two 1-cocycles for the fpqc-cover $p\colon X' \to X$, say $g^{(1)}$ and $g^{(2)}$. Thus $\widetilde{g}^{(i)}$, $i = 1, 2$, are descent data. An isomorphism between between these two descent data is an automorphism of X'-schemes $\varphi\colon G_{X'} \to G_{X'}$ such that $p_1^*(\varphi)\widetilde{g}^{(1)} = \widetilde{g}^{(2)}p_2^*(\varphi)$. If E_1 and E_2 are the two torsors given by $\widetilde{g}^{(1)}$ and $\widetilde{g}^{(2)}$, then φ gives rise to an isomorphism of X-schemes (not necessarily of G-torsors) $\bar{\varphi}\colon E_1 \xrightarrow{\sim} E_2$. For $\bar{\varphi}$ to be an isomorphism of G-torsors it is necessary and sufficient that $\varphi = \widetilde{t}$ for some $t \in G(X')$, for then and only then is φ a G-equivariant isomorphism (i.e., an automorphism of the G-torsor $G_{X'}$. The conclusion is that $g^{(i)}\colon X'' \to G$, $i = 1, 2$, give rise to isomorphic G-torsors if and only if there exists $t \in G(X')$ such that the diagram below commutes (compare with (1.1.3)):

$$\begin{array}{ccc} G_{X''} & \xrightarrow[\sim]{\widetilde{g}^{(1)}} & G_{X''} \\ {\scriptstyle p_2^*(\widetilde{t})}\downarrow\wr & & \wr\downarrow{\scriptstyle p_1^*(\widetilde{t})} \\ G_{X''} & \xrightarrow[\widetilde{g}^{(2)}]{\sim} & G_{X''} \end{array}$$

and this happens if and only if there exists $t \in G(X')$ such that for a valued point $(x_1', x_2')\colon W \to X''$ of X'', the following equation is satisfied:

$$t(x_1')g^{(1)}(x_1',\, x_2') = g^{(2)}(x_1',\, x_2')t(x_2'). \tag{1.2.1}$$

1.3. **The set** $\mathrm{H}^1(X,\, G_X)$**.** The above considerations lead us to put an equivalence relation on the set of 1-cocycles in $G(X'')$, i.e., namely the relation $g^{(1)} \sim g^{(2)}$ if there exists $t \in G(X')$ such that (1.2.1) is satisfied for all valued points (x_1', x_2', x_3') of X'''. The resulting set of equivalence classes is denoted $\check{\mathrm{H}}^1(X',\, G_X)$. In other words:

$$\check{\mathrm{H}}^1(X',\, G_X) = \{g \in G(X'') \,|\, g(p_{12})g(p_{23}) = g(p_{13})\}/\sim .$$

From our discussion above, the elements of $\check{\mathrm{H}}^1(X',\, G_X)$ are in one-to-one correspondence with isomorphism classes of G-torsors on X which are trivialised over X'. For a fixed locally quasi-affine G- space F, elements of $\check{\mathrm{H}}^1(X',\, G_X)$ are also in one-to-one correspondence with isomorphism classes of (G, F)-spaces over X (under (G, F)-morphisms), which are trivialised over X'.

If $u\colon T \to X'$ is an fpqc map, then clearly we have a map $u^*\colon \check{\mathrm{H}}^1(X',\, G_X) \to \check{\mathrm{H}}^1(T,\, G_X)$, induced by $g \mapsto g(u \times u)$, where $u \times u\colon T \times_X T \to X''$ is the obvious map induced by u.

If one could take the direct limit of the sets $\check{\mathrm{H}}^1(X',\, G_X)$ as X' runs through fpqc covers of X, then the direct limit set would (obviously) be in bijective correspondence with isomorphism classes of G-torsors over X. Here is where we run into a logical difficulty, for the class of fpqc-maps are much too large and direct limits will not exist unless a universe is fixed (so that the class of fpqc-maps $X' \to X$ can essentially be regarded as a set). Fixing a universe in this case is not very helpful because the direct limit is depends on the universe used. However the class of étale surjective maps $X' \to X$ is essentially small (as is the class of fppf maps to X), and one can take direct limits over these. The resulting direct limit will be in bijective correspondence with isomorphism classes of G-torsors over X. This direct limit is denoted $\mathrm{H}^1(X,\, G_X)$ (note the absence of a "check" over H^1). The direct limit could also have been taken over fppf maps to get the same limiting set.

The set $\mathrm{H}^1(X,\, G_X)$ is called the *first cohomology set with coefficients in the sheaf* G_X. Here G_X is regarded as a sheaf over X for the étale-topology. It can also be regarded as a sheaf for the fppf-topology.

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