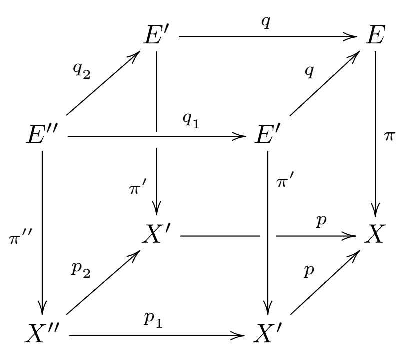
LECTURE 18

1. $E(F)$ as a quotient by an equivalence relation

We now give the proof of the statement made in Lectures 16 and 17 regarding $E(F)$ as a quotient of a trivial fibre space by an fpqc-equivalence relation.

1.1. **Cartesian cubes.** Suppose $\pi\colon E \to X$ is a G-torsor, and $p\colon X' \to X$ is an fpqc-map such that there is a trivialization $\theta\colon G_{X'} \xrightarrow{\sim} E_{X'}$. For simplicity, write $E' = E_{X'}$, $E'' = E_{X''}$ and let $\pi'\colon E' \to X'$ and $\pi''\colon E'' \to X''$ the natural maps induced by π. We have a cartesian cube, with the maps q, q_1, and q_2 the obvious base changes of p, p_1, and p_2 respectively.

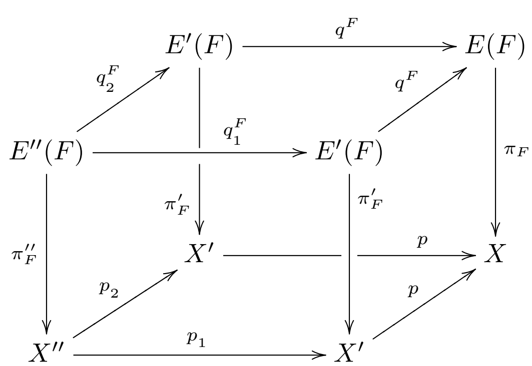
(1.1.1)



Note that the diagram summarizes a lot of data, including the identities $E' = p^*E$ and $E'' = p_1^*E' = p_2^*E'$.

If $F \to S$ is a locally quasi-affine G-space over S we then have a commutative cartesian cube, analogous to—and arising from—diagram (1.1.1), namely

(1.1.2)



with q^F and q_i^F being the natural projections.

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Let us re-write the top face, namely the cartesian square:

$$\begin{array}{ccc} E''(F) & \xrightarrow{q_2^F} & E'(F) \\ {\scriptstyle q_1^F}\downarrow & \square & \downarrow{\scriptstyle q^F} \\ E'(F) & \xrightarrow[q^F]{} & E(F) \end{array}$$

It is evident that if $\xi_1 = (x'_1, w_1)$ and $\xi_2 = (x'_2, w_2)$ are T-valued points of $X' \times_X E(F) = E'(F)$, then a necessary and sufficient condition for (ξ_1, ξ_2) to be a T-valued point of $X'' \times_X E(F) = E''(F)$ is: $p(x'_1) = p(x'_2)$ and $w_1 = w_2$. Thus if we regard cartesian squares with fppf-arrows as effective equivalence relations, then the induced equivalence relation $\mathfrak{R}$ on $E'(F)$ is $(x'_1, w_1)\mathfrak{R}(x'_2, w_2)$ if and only if $p(x'_1) = p(x'_2)$ and $w_1 = w_2$.

Now let $\theta_F \colon F_{X'} \xrightarrow{\sim} E'(F)$ be the trivialization of the F-fibre bundle induced by θ so that the automorphism ψ_{12} of the X''-scheme $F_{X''}$ given by

$$\psi_{12} := p_1^*(\theta_F)^{-1} \circ p_2^*(\theta_F)$$

is the one characterised by

$$(x'', f) \mapsto (x'', g_\theta(x'')f)$$

for T-valued point x'' and f of X'' and F respectively. Here $g_\theta \colon X'' \to G$ the transition function determined by θ. As in Lecture 17, for $i = 1, 2$ let $r_i = \theta_F^{-1} \circ q_i^F$. We then have a cartesian square

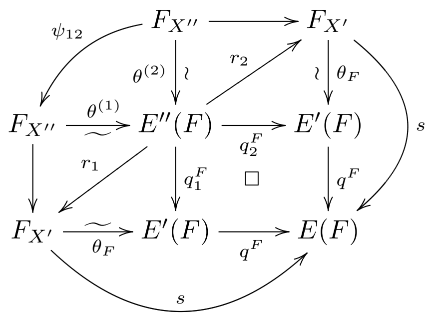
$$\begin{array}{ccc} E''(F) & \xrightarrow{r_2} & F_{X'} \\ {\scriptstyle r_1}\downarrow & \square & \downarrow{\scriptstyle s} \\ F_{X'} & \xrightarrow[s]{} & E(F) \end{array}$$

This is equivalent to regarding $E(F)$ as the quotient of $F_{X'}$ by an equivalence relation, namely the equivalence relation induced by the equivalence relation $\mathfrak{R}$ on $E'(F)$ and the isomorphism $\theta_F \colon F_{X'} \xrightarrow{\sim} E'(F)$. Let this equivalence be denoted $\sim_\theta$. Then $(x'_1, f_1) \sim_\theta (x'_2, f_2)$ if and only if $\xi_1 \mathfrak{R}\, \xi_2$, where $\xi_i = \theta_F(x'_i, f_i)$. Note that this happens if and only if we have an valued point w of $E(F)$ such that $\theta_F(x'_i, f_i) = (x'_i, w)$ for $i = 1, 2$ and $p(x'_1) = p(x'_2)$. Consider the valued point $\xi^* = (x'_1, x'_2, w)$ of $E''(F)$. Clearly $p_i^*(x'_1, x'_2, f_i) = \xi^*$ for $i = 1, 2$. Thus $(x'_1, f_1) \sim_\theta (x'_2, f_2)$ if and only if $p(x'_1) = p(x'_2)$ and $(x'_1, x'_2, f_1) = \psi_{12}(x'_1, x'_2, f_2)$. We have thus shown that

$$(x'_1, f_1) \sim_\theta (x'_2, f_2) \Longleftrightarrow p(x'_1) = p(x'_2) \text{ and } f_1 = g_\theta(x'_1, x'_2) f_2$$

From earlier argument, this also means that $E(F) = E \times_S F/\sim$ where $(e, f) \sim (eg, g^{-1}f)$ for $e \in E(T)$, $f \in F(T)$ and $g \in G(T)$, and $T \in \mathbb{S}\mathrm{ch}_{/S}$.

The following commutative diagram may help with book-keeping (where, for $i = 1, 2$, we write $\theta^{(i)} = p_i^*(\theta)$.



We have proved:

Theorem 1.1.3. *Let $\pi\colon E \to X$ be a G-torsor and $F \to S$ a locally quasi-affine G-space.*

(a) *Let $(X' \xrightarrow{p}, \theta)$ be a trivialization of E and $\sim_\theta$ the equivalence relation on $F_{X'}$ given by*

$$(x_1', f_1) \sim_\theta (x_2', f_2) \iff p(x_1') = p(x_2') \text{ and } f_1 = g_\theta(x_1', x_2')f_2.$$

Then

$$E(F) = F_{X'}/\sim_\theta .$$

(b) *If $\sim$ is the equivalence relation on $E \times_S F$ given by $(e, f) \sim (eg, g^{-1}f)$ for $e \in E(T)$, $f \in F(T)$ and $g \in G(T)$, and $T \in \mathbb{S}ch_{/S}$, then*

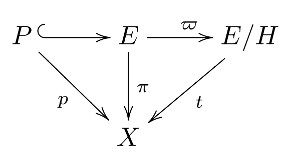
$$E(F) = E \times_S F/\sim .$$

2. Reductions revisited

Let us return to the situation of Hausdorff topological spaces, with G a topological group and H a closed subgroup.

Let $\pi\colon E \to X$ be a principal G-bundle. We saw last time that (isomorphism classes of) reductions of structure group of E to H are in bijective correspondence with principal H-sub-bundles of E. Indeed, if $P \hookrightarrow E$ is such a principal H-sub-bundle and $p = \pi|_P\colon P \to X$, then the associated fibre bundle $P(G)$ with fibre G consists of equivalence classes of pairs $(a, g) \in P \times G$ with $(a, g) \sim (ah, h^{-1}, g)$ for $h \in H$. If $[a, g]$ is the equivalence class of (a, g), then the map $[a, g] \mapsto ag$ is a well defined continuous map $\varphi\colon P(G) \to E$ lying over X. Conversely, given $e \in E$, the intersection $P \cap \pi^{-1}(\pi(e))$ is non-empty, and hence we can pick $a \in P \cap \pi^{-1}(\pi(e))$. There is a unique $g \in G$ such that $ag = e$. The point $\psi(e) := [a, g]$ does not depend on the choice of $a \in P \cap \pi^{-1}(\pi(e))$ and we have a well-defined continuous map $\psi\colon E \to P(G)$ given by $e \mapsto \psi(e)$. The maps φ and ψ are inverses.

So suppose P is as above. We have a commutative diagram



Let us write $P_x = p^{-1}(x)$ and $E_x = \pi^{-1}(x)$ for $x \in X$. Identifying E_x with G (this is tantamount to picking a point $e \in E_x$), P_x identifies as a coset of H. Thus if

$a \in P_x$, then $P_x := P_x = aH$. It follows that the image $\varpi(P_x) \subset E/H$ of P_x in E/H is a single point, say $\sigma_P(x)$. As x varies in X, the fibres P_x vary in a continuous fashion, whence so do the point $\sigma_P(x)$. The assignment $x \mapsto \sigma_P(x)$ is therefore a continuous section of $t\colon E/H \to X$. Conversely, given a section σ of $t\colon E/H \to X$, as x varies in X, we have subspaces $P_x = \varpi^{-1}(\sigma(x))$ of G_x varying continuously with x. It is not hard to see that the (disjoint) union $P = \cup_x P_x$ principal H-subbundle of E. This gives us another way of thinking about the continuous sections of t arising from reductions of structure groups.

2.1. **Sections of $E(F)$ and equivariant maps.** Let X be a topological space and F a G-space. Let $\pi\colon E \to X$ be a principal G-bundle. Recall that one description of the fibre bundle $E(F)$ is as the quotient $(E \times F)/G$ where the action of G on $E \times F$ is the right action given by $(e,\, f) \cdot g = (eg,\, g^{-1}f)$. Note that we have a natural map $\pi_F\colon E(F) \to X$. Suppose $\sigma\colon X \to E(F)$ is a section of π_F. Let $e \in E$ and let $x = \pi(e)$. The element $\sigma(x)] in E(F)$ must be of the form $[e^*,\, f^*]$ for some $e^* \in E$ with $\pi(e^*) = x$ and some $f^* \in F$. Since $\pi(e) = x = \pi(e^*)$, there is a unique $g \in G$ such that $e^* = eg$, whence $[e^*,\, f^*] = [e,\, gf]^*$. Setting $f = f^*$ we see that we may always write $\sigma(\pi(e))$ in the form $[e, f]$ for a suitable $f \in F$. Moreover this element $f \in F$ is unique. Indeed if $[e,\, f] = [e,\, f_1]$, then $(e,\, f) = (eg,\, g^{-1}f_1)$ for a suitable $g \in G$. This means $e = eg$, which in turn means $g = 1$, for G acts freely on E. Thus $f = f_1$. Thus $f \in F$ completely determined by $e \in E$ and the section σ as the unique element such that $\sigma(\pi(e)) = [e, f]$. This unique f may therefore be written as $f = \varphi_\sigma(e)$. The assignment $e \mapsto \varphi_\sigma(e)$ gives a map

$$\varphi_\sigma\colon E \to F$$

and it is characterized by the formula $\sigma \circ \pi(e) = [e,\, \varphi_\sigma(e)]$ for $e \in E$. Now

$$[eg,\, g^{-1}\varphi_\sigma(e)] = [e,\, \varphi_\sigma(e)] \qquad (e \in E, g \in G)$$

whence,

$$\varphi_\sigma(eg) = g^{-1}\varphi_\sigma(e) \qquad (e \in E, g \in G).$$

This means $\varphi_\sigma\colon E \to F$ is G-equivariant for the *right* G-action on F.

Conversely, if $\varphi\colon E \to F$ is G-equivariant, i.e., $\varphi(eg) = g^{-1}\varphi(e)$ for $e \in E$ and $g \in G$, then the element $[e,\, \varphi(e)]$ depends only on the image $x = \pi(e)$ and not on e. Indeed, if $e' \in E$ is another point on the fibre of π over x, then $e' = eg$ for a unique $g \in G$, and $\varphi(e') = \varphi(eg) = g^{-1}\varphi(e)$. Thus $[e',\, \varphi(e')] = [eg,\, g^{-1}\varphi(e)] = [e,\, \varphi(e)]$. Thus the map $E \to E(F)$ given by $e \mapsto [e,\, \varphi(e)]$ factors through X, and we get a map

$$\sigma_\varphi\colon X \to E(F)$$

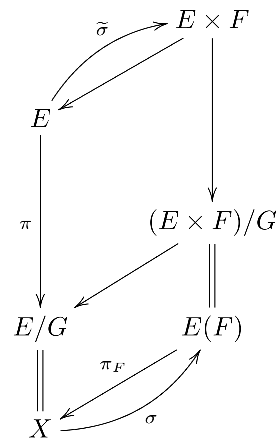
given by

$$x \mapsto [e,\, \varphi(e)] \qquad (x \in X)$$

where e is any element of $\pi^{-1}(x)$.

Perhaps the most conceptual way of seeing this correspondence between sections of $\pi_F\colon E(F) \to X$ and G-equivariant maps $G \to F$ is as follows. As we noted above, G acts on the right on $E \times F$ by the formula $(e,\, f) \cdot g = (eg,\, g^{-1}f)$ for $(e,\, f) \in G \times F$ and $g \in G$, and $E(F) = (E \times F)/G$. Further $X = E/G$. Clearly, a G-equivariant section of $E \times F \to E$ descends to a section of $(E \times F)/G \to E/G$. Conversely, every section of $(E \times F)/G \to E/G$ gives rise to a G-equivariant section of $E \times F \to E$, for $E \times F = E \times_{E/G} ((E \times F)/G)$. However, G-equivariant sections

$\widetilde{\sigma}\colon E \to E \times F$ of $E \times F \to E$ are exactly the maps $(1_E, \varphi)$ where $\varphi\colon E \to F$ is G-equivariant.



Whichever way we look at it, we end up with the following:

Lemma 2.1.1. *There is a bijective correspondence between the sections of $E(F)$ over X and G-equivariant maps $E \to F$ for the right actions of G on E and F.*

Remarks 2.1.2. (1) If $\varphi\colon E \to G/H$ is G-equivariant (therefore giving rise to a unique section of $E/H \to X$, then the corresponding H-sub-bundle P of $\pi\colon E \to X$ can be obtained directly by setting $P = \varphi^{-1}(\xi_0)$ where ξ_0 is the distinguished point of G/H, namely the image of the identity element $\varepsilon \in G$ under the natural map $G \to G/H$. Since φ is G-equivariant, and since ξ_0 is H-invariant, clearly P is H-stable. It is not hard to see that $P \to X$ is in fact a principal H bundle and the inclusion of P into E is H-equivariant.

(2) If X is a differential manifold and $E^T \to X$ the principal $GL_n(\mathbf{R})$-bundle associated to the tangent bundle $T_X \to X$, then a Riemannian structure on X can be re-interpreted as a reduction of structure group of E^T to $SO_n(\mathbf{R})$. In particular, Riemannian structures on X are in bijective correspondence with sections $\sigma\colon X \to E^T/SO_n(\mathbf{R})$ of the fibre bundle $E^T/SO_n(\mathbf{R}) \to X$.

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