LECTURE 17

1. $E(F)$ as the quotient of $E \times_S F$ by an equivalence relation

1.1. As before, let $\pi\colon E \to X$ be a G-torsor, and $F \to S$ a locally quasi-affine G-space, and $F_{X'} := X' \times_S F$. The last time we stated without proof that $E(F)$ is realisable as $F_{X'}/R_\theta$ where $X' \to X$ is a trivialising fpqc cover of $E \to X$, $\theta\colon G_{X'} \xrightarrow{\sim} E_{X'}$ a trivialisation, and R_θ given by the equivalence relations

$$((x'_, f_1)\, R_\theta(T)\, (x'_2,\, f_2)) \Longleftrightarrow p(x'_1) = p(x'_2) \text{ and } f_1 = g_\theta(x'_1,\, x'_2) f_2. \tag{1.1.1}$$

Let us postpone the (easy) proof to the next lecture. However note that if $X' = E$ with the trivialising cover $X' \to X$ being $E \xrightarrow{\pi} X$, and if the trivialsation θ is given by the orbit $\psi = \psi_\delta$ map of the diagonal section $\delta\colon E \to E \times_X E$, i.e. the map

$$\psi\colon (e, g) \mapsto (e,\, eg)$$

from G_E to $E \times_X E$, then the above description shows that

$$E(F) = (E \times_S F)/\sim \tag{1.1.2}$$

where the equivalence relation $\sim$ is given by $(e,\, f) \sim (eg,\, g^{-1}f)$ for appropriate e, f, and g. Indeed for two T-valued point $e, e^*\colon T \rightrightarrows E$, $e^* = eg$ for some $g \in G(T)$ if and only if $\pi(e) = \pi(e^*)$ and in this case g is unique. In fact, $p(e) = p(e^*)$ implies that $e^* = eg_\psi(e,\, e^*)$ and this defines the unique g such that $e^* = eg$. Thus, we have the equality

$$g_\psi(e,\, eg) = g$$

where ψ is the "diagonal trivialisation" on $E \times_X E \to E$ (via the first projection). It follows that on $(E \times_S F)(T)$ we have $(e,\, f) \sim (e^*,\, f^*)$ if and only if $e^* = eg$ for some $g \in G(T)$, necessarily unique, and for this g, $f = gf^*$. Thus $(e,\, gf^*) \sim (eg,\, f^*)$, or, equivalently, $(e,\, f) \sim (eg,\, g^{-1}f)$.

1.2. Let $\pi\colon E \to X$ and $F \to S$ be as above. To lighten notation, write $E' = E_{X'}$ and $E'' = E_{X''}$ and let the respective maps to X' and X'' be $\pi'\colon E' \to X'$ and $\pi''\colon E'' \to X''$. With $F \to S$ as above, we have associated fibre spaces $\pi_F\colon E(F) \to X$, $\pi'_F\colon E'(F) \to X'$, and $\pi''_F\colon E''(F) \to X''$. It is immediate (from the construction of $E(F)$ via the trivialization θ) that $E'(F) = X' \times_X E(F)$ and $E''(F) = X'' \times_X E(F)$ and that under these identifications, $\pi'_F = X' \times_X \pi_F$, and $\pi''_F = X'' \times_X \pi_F = X'' \times_{X'} \pi'_F$. For $i = 1, 2$, let

$$q_i^F\colon E''(F) \to E'(F)$$

be the base change of $p_i\colon X'' \to X'$ under the map $\pi'_F\colon E'(F) \to X'$, and let

$$q^F\colon E'(F) \to E(F)$$

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be the base change of $p\colon X' \to X$ by the map $\pi\colon E(F) \to X$. We then have a cartesian square

$$\begin{array}{ccc} E''(F) & \xrightarrow{q_2^F} & E'(F) \\ {\scriptstyle q_1^F}\downarrow & \square & \downarrow{\scriptstyle q^F} \\ E'(F) & \xrightarrow[q^F]{} & E(F) \end{array}$$

Next, recall we have an isomorphism

$$\theta_F\colon F_{X'} \xrightarrow{\sim} E'(F)$$

of fibre-spaces over X' with structure group G such that the induced automorphism

$$\psi_{12} := p_1^*(\theta_F)^{-1} \circ p_2^*(\theta_F)$$

of the X''-schemes $F_{X''}$ is given by

$$(x'',\, f) \mapsto (x'',\, g_\theta(x'')f).$$

For $i = 1, 2$, let

$$r_i\colon E''(F) \to F_{X'}$$

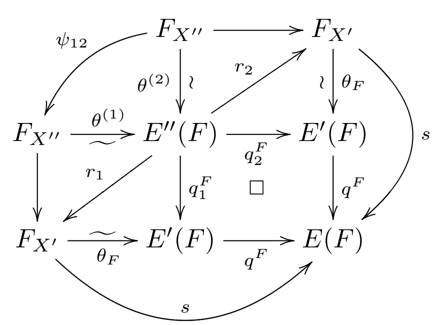
be the map $r_i = \theta_F^{-1} \circ q_i^F$ and

$$s\colon F_{X'} \to E(F)$$

the natural map arising from the construction of $E(F)$ (by making $F_{X'}$ "descend" from X' to X along $p\colon X' \to X$). Note $s = q^F \circ \theta_F$. We therefore have a cartesian diagram

$$\begin{array}{ccc} E''(F) & \xrightarrow{r_2} & F_{X'} \\ {\scriptstyle r_1}\downarrow & \square & \downarrow{\scriptstyle s} \\ F_{X'} & \xrightarrow[s]{} & E(F) \end{array}$$

Thus $E''(F)$ is an equivalence relation on $F_{X'}$. The two cartesian diagrams above fit into the following expanded commutative diagram, in which we write $\theta^{(i)}$ as a shorthand for the typographically inconvenient symbol $p_i^*(\theta_F)$, for $i = 1, 2$.



Using this, it is not hard to see that if $s(x_1',\, f_1) = s(x_2',\, f_2)$ for two valued points $(x_1',\, f_1)$ and $(x_2',\, f_2)$ of $F_{X'}$ with the same source, (i.e., the two points determine a unique point of $E''(F)$) then (a) $p(x_1') = p(x_2')$ and (b) $f_1 = g_\theta(x_1',\, x_2')f_2$. Conversely, if $(x_1',\, f_1)$ and $(x_2',\, f_2)$ (valued points of on $F_{X'}$ with the same source) satisfy (a) and (b), then necessarily $s(x_1',\, f_1) = s(x_2',\, f_2)$. This just means that $E''(F)$ represents the equivalence the functor R_θ, whence R_θ is a scheme-theoretic equivalence relation. Since the map $s\colon F_{X'} \to E(F)$ is a fpqc, it follows from earlier discussions that $E(F) = F_{X'}/R_\theta$. We will give details of the proof of $R_\theta = E''(F)$ in the next lecture.

2. Principal Bundles and the space $E(G/H)$

Let us return to the topological situation. So, G is a topological group and all topological spaces (including G) in this section are Hausdorff, and all actions of G on topological spaces are continuous.

2.1. The principal bundle $G \to G/H$. Suppose H is a closed subgroup of G. The space G/H has a natural structure of a topological space, via the quotient topology induced from G and the natural map $p_H\colon G \to G/H$. This means that G/H has the coarsest topology which makes $p_H\colon G \to G/H$ continuous. In even simpler terms, a subset $S \subset G/H$ is an open subset if and only if $p^{-1}(S)$ is open in G, and this defines a topology on G/H. Recall that a continuous map $f\colon V \to W$ is said to have *local sections* if each point $w \in W$ has an open neighborhood U and a continuous section of the map $f^(U) \to U$ induced by restricting f to $f^1(U)$.

Lemma 2.1.1. *The continuous map $p_H\colon G \to G/H$ is H-equivariant for the right action of H on G and the trivial action of H on G/H. Moreover, $p_H\colon G \to G/H$ is a principal H-bundle with this right action of H on G, if and only if p_H has local sections.*

Proof. Every principal bundle has local sections, for trivialisations are equivalent to sections. Thus we only have to prove that if $p_H\colon G \to G/H$ has local sections, then it is a principal H-bundle. Suppose $\mathscr{U} = \{U_\alpha\}$ is a cover of X and for each index α we have continuous sections $\sigma_\alpha\colon U_\alpha \to p_H^{-1}(U_\alpha)$ of the map $p_H^{-1}(U_\alpha) \to U_\alpha$ induced by α. For each α the map

$$U_\alpha \times H \xrightarrow{\varphi_\alpha} p_H^{-1}(U_\alpha)$$

given by

$$(u,\, h) \mapsto \sigma_\alpha(u)h$$

is clearly a bijective map at the level of sets. It is continuous because σ_α is continuous and multiplication in G is continuous on $G \times G$. To see the inverse is continuous, we examine the inverse more closely. If $g \in p_H^{-1}(U_\alpha)$, and we write $u = p_H(g)$ then g and $\sigma(u)$ lie on the same fibre of p, namely the coset gH. It follows that there is a unique h ($=h_\alpha(g)$) in H such that $\sigma_\alpha(u)h = g$. The inverse map is then $g \mapsto (p_H(g),\, h_\alpha(g))$. It is clear that $g \mapsto h_\alpha(g)$ is a continuous map from $p_H^{-1}(U_\alpha)$ to H. Indeed $h_\alpha(g) = (\sigma_\alpha(p(g)))^{-1}g$, exhibiting h_α as a composite of continuous maps on G. This gives local trivialisations. Since $p_H\colon G \to G/H$ is H-equivariant and has local trivialisations, we are done by Problem (12) of your Mid-Term exam. □

Remark 2.1.2. It may be instructive to work out the 1-cocyles arising from the σ_α in the proof of the lemma. For two indices α and β, define

$$h_{\alpha\beta}\colon U_{\alpha\beta} \to H$$

by

$$h_{\alpha\beta}(u) = \sigma_\alpha(u)^{-1}\sigma_\beta(u)$$

for $u \in U_{\alpha\beta}$. Again, since $\sigma_\alpha(u)$ and $\sigma_\beta(u)$ lie in the same coset of H (recall, fibres of p are cosets of H), therefore $h_{\alpha\beta}$ does indeed take values in H. Moreover, it is continuous since taking inverses and multiplication are continuous operations. Obviously $(h_{\alpha\beta})$ is a 1-cocycle.

2.2. The space $E(G/H)$ and the space E/H. Let $\pi\colon E \to X$, and H be as above, and assume $p_H\colon G \to G/H$ has local sections. Now G acts on G/H and hence we get the associated fibre bundle $\varpi\colon E(G/H) \to X$. Clearly if E is the trivial principal bundle $X \times G$, then $E(G/H) = X \times (G/H)$ by construction of $E(F)$ via 1-cocyles. However, $X \times (G/H) = (X \times G)/H$. Thus $E(G/H) = E/H$ in this case. In the general case, suppose $\mathscr{U} = \{U_\alpha\}$ is an open cover and we have G-trivialisations

$$\varphi_\alpha\colon G_{U_\alpha} \xrightarrow{\sim} \pi^{-1}(U_\alpha) = U_\alpha \times_X E$$

giving rise to the 1-cocycle $(g_{\alpha\beta})$. Then

$$E = \coprod_\alpha (U_\alpha \times G)/R$$

for the equivalence relation R induced by $(g_{\alpha\beta})$. If $X' = \coprod_\alpha U_\alpha$, then the principal G-bundle $E_{X'} \to X'$ is a trivial torsor, and $E_{X'}(G/H)$ is therefore isomorphic to $X' \times (G/H) = (X' \times G)/H$. The 1-cocycle $(g_{\alpha\beta})$ gives rise to an equivalence R_H relation on $(X' \times G)/H$ and $E(G/H)$ has a realisation as $(X' \times G/H)/R_H$. Since both R and R_H arise from $(g_{\alpha\beta})$ and the left action of G on $X' \times G$ commutes with the right action of G, it is not hard to see that the following diagram commutes

$$\begin{array}{ccc}
\coprod_\alpha U_\alpha \times G & \xrightarrow{\text{modulo } R} & E \\
{\scriptstyle \text{modulo } H}\downarrow & & \downarrow{\scriptstyle \text{modulo } H} \\
(\coprod_\alpha U_\alpha \times G)/H & \xrightarrow{(\text{modulo R})/H} & E/H \\
\| & & \| \\
\coprod_\alpha U_\alpha \times (G/H) & \xrightarrow[\text{modulo } R_H]{} & E(G/H)
\end{array}$$

The net result is that we have $E(G/H) = E/H$.

There is a more canonical way of seeing this which is perhaps more illustrative. Let $\varepsilon \in G$ be the identity element, and $\xi_0 \in G/H$ the image of ε under the natural map $p\colon G \to G/H$. We have a natural continuous map

$$\varpi\colon E \to E(G/H)$$

given by

$$e \mapsto [e,\, \xi_0].$$

Note that $[e, \xi_0] = [e', \xi_0]$ if and only if there is an element (necessarily unique) $g \in G$ such that $(e', \xi_0) = (eg, g^{-1}\xi_0)$. This forces the equation $g\xi_0 = \xi_0$, i.e., $g \in H$. Thus $\varpi(e) = \varpi(e')$ if and only if they are in the same H-orbit. It follows that

$$E(G/H) = E/H \tag{2.2.1}$$

and under this identification $\varpi\colon E \to E(G/H)$ is the quotient map $E \to E/H$. Let

$$t\colon E/H \to X$$

denote the resulting fibre bundle with fibre G/H. We then have a commutative diagram

(2.2.2)
$$\begin{array}{ccc} E & \xrightarrow{\varpi} & E/H \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle t} & \\ & X & \end{array}$$

Proposition 2.2.3. *Suppose* $p_H \colon G \to G/H$ *has* local sections. *The space* E *is a principal* H*-bundle over* $E(G/H) = E/H$.

Proof. Clearly $\varpi \colon E \to E/H$ is H-equivariant for the trivial H-action on E/H. Let $\mathscr{W} = \{W_i\}$ be a trivializing cover for the principal H-bundle $p_H \colon G \to G/H$. Let $\mathscr{U} = \{U_\alpha\}$ be a trivializing cover for the principal G-bundle $\pi \colon E \to X$. Then $\mathscr{U}$ is a also a trivializing cover for the fibre-bundle $t \colon E/H \to X$. It follows that on $t^{-1}(U_\alpha)$ we can find an open cover $\mathscr{V} = \{V_{\alpha i}\}$ such that each $V_{\alpha i}$ is homeomorphic to $U_\alpha \times W_i$, and $\varpi^{-1}(V_{\alpha i}) \to V_{\alpha i}$ is a trivial principal H-bundle. From Problem (12) of the mid-term, the result follows. □

3. Reduction of structure groups

Let $\pi \colon E \to X$ be a principal G-bundle and H a closed subgroup of G. Recall that this principal G-bundle is said to have a reduction of structure group to H if there is an open cover $\mathscr{U} = \{U_\alpha\}$ and 1-cocycle $(h_{\alpha\beta})$ for $E \to X$ with respect to $\mathscr{U}$ such that the $h_{\alpha\beta}$ take values in H.

There is another way of viewing this. If $\mathscr{H}$ is the sheaf $U \mapsto \{U \xrightarrow{\psi} H \mid \psi \text{ is continuous}\}$ and $\mathscr{G}$ the sheaf $U \mapsto \{U \xrightarrow{\varphi} G \mid \varphi \text{ is continuous}\}$ then we have a natural map

$$h_* \colon \mathrm{H}^1(X, \mathscr{H}) \to \mathrm{H}^1(X, \mathscr{G})$$

for every 1-cocycle with values in H is a 1-cocycle with values in G and if two of them are H-cohomologous, then they are G-cohomologous. Let $\zeta \in \mathrm{H}^1(X, \mathscr{G})$ be the element defined by the isomorphism class of E. The structure group of E is *reducible to* H is and only if ζ is in the image of h_*. In this case, a specific choice of an element $\xi \in \mathrm{H}^1(X, \mathscr{H})$ such that $h_*\xi = \zeta$ is a *reduction of structure group of* E *to* H.

Assumption: From now on we assume that the map $G \to G/H$ has *local sections*.

3.1. Suppose we are given a 1-cocycle $(h_{\alpha\beta})$ for $E \to X$ with respect to $\mathscr{U}$ such that the $h_{\alpha\beta}$ take values in H. We can construct a principal H-bundle

$$p \colon P \to H$$

from our cocycle, and we have a natural H-equivariant inclusion of a closed subspace $i \colon P \hookrightarrow E$ such that $\pi \circ i = p$:

(3.1.1)
$$\begin{array}{ccc} P & \overset{i}{\hookrightarrow} & E \\ & {\scriptstyle p} \searrow & \downarrow {\scriptstyle \pi} \\ & & X \end{array}$$

In fact the natural inclusion $\coprod_\alpha U_\alpha \times H \hookrightarrow \coprod_\alpha U_\alpha \times G$ is H-equivariant for the right and left actions, and since P and E are obtained by quotienting by the same cocycle acting on the left, the right action survives the compatible quotienting processes and gives $i\colon P \hookrightarrow E$.

Conversely, given a commutative diagram as above with $i\colon P \to E$ an H-equivariant inclusion map, identifying P as a H-stable closed subspace of E, we can find local sections $\sigma_\alpha\colon U_\alpha \to p^{-1}(U_\alpha)$ of $p^{-1}(U_\alpha) \to U_\alpha$ and maps

$$h_{\alpha\beta}\colon U_{\alpha\beta} \to H$$

satisfying

$$\sigma_\beta(u) = \sigma_\alpha(u)h_{\alpha\beta}(u) \qquad (u \in U_{\alpha\beta}).$$

Then $(h_{\alpha\beta})$ is a 1-cocycle. Now $i \circ \sigma_\alpha = \tau_\alpha$ (say) give rise to local sections of $\pi\colon E \to X$, and since i is H-equivariant, the corresponding cocycle is precisely $(h_{\alpha\beta})$. In other words we have a reduction of the structure group G to H.

There is yet another way of seeing that principal H-sub-bundles $p\colon P \to X$ of $\pi\colon E \to X$ give a reduction of structure group of E to H. Note that by H-sub-bundle we mean that $p\colon P \to X$ fits into the diagram (3.1.1) with $i\colon P \to E$ H-equivariant and an inclusion of a closed subspace. Now H acts on G from the left by multiplication, and since this action commutes with the right action of G on G, the resulting associated fibre bundle $P(G)$ has a right G-action on it, and hence is necessarily a principal G-bundle since it is already locally trivial (using the trivialisations of $P \to X$). In fact the transition functions for $P(G)$ are the same as that for P and hence $P(G)$ has a reduction of structure group to H. We shall show that $P(G) \to X$ and $\pi\colon E \to X$ are isomorphic as principal G-bundles. That will show that H-sub-bundles like P give rise to reductions of structure groups. To that end, let $[a, g] \in P(G)$. Recall $[a, g] = [a', g']$ if and only if there is an $h \in H$ such that $(a', g') = (ah, h^{-1}g)$. Define

$$P(G) \to E$$

by

$$[a, g] \mapsto i(a)g.$$

This is clearly well defined (since i is H-equivariant), G-equivariant, and continuous. The inverse map is as follows. Let $e \in E$ and set $x = \pi(e)$. Let $a \in P$ be any element in the fibre of x in P. (Note that if $a' \in P$ is another element such that $p(a') = x$ then there is a unique $h \in H$ such that $a' = ah$.) Let $g \in G$ be the unique element such that $e = i(a)g$. The element $[a, g] \in P(E)$ does not depend on the choice of a in $p^{-1}(x)$, for by the H-equivariance of i, we have $e = i(ah)h^{-1}g$ for $h \in H$, and $[a, g] = [ah, h^{-1}g]$. The map

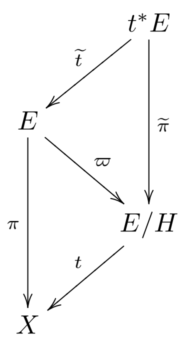
$$E \to P(E)$$

given by $e \mapsto [a, g]$, with a and g as above, is continuous, and gives the required inverse. Thus

$$P(G) \xrightarrow{\sim} E.$$

3.2. A particular reduction of structure group. Diagram (2.2.2) can be expanded to

(3.2.1)



where t^*E is a short-hand for $E/H \times_X E$ and $\widetilde{t}$ and $\widetilde{\pi}$ are the projections. We have a map

$$\widetilde{\sigma}_0 \colon E \to t^*E$$

given by $e \mapsto (\varpi(e), e)$. This is a section, and hence identifies E as a closed subspace of t^*E. Clearly $\widetilde{\pi} \circ \widetilde{\sigma}_0 = \varpi$. Further, for $h \in H$ and $e \in E$

$$\widetilde{\sigma}_0(eh) = (\varpi(eh), eh) = (\varpi(e), , eh) = (\varpi(e), e) \cdot h = (\widetilde{\sigma}_0(e))h.$$

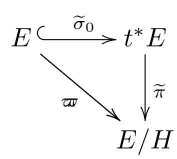
Thus $\widetilde{\sigma}_0$ is H-equivariant. Recall that reductions of structure groups are characterized by diagrams such as (3.1.1). Thus $E \to E/H$ is a principal H-sub-bundle of t^*E, thus giving a reduction of structure group of $\widetilde{\pi} \colon t^*E \to E/H$ to H.

Theorem 3.2.2. *Let H be a closed subgroup of G such that the canonical map $p_H \colon G \to G/H$ has local sections. With notations as above, reductions of structure group of the principal G-bundle E to H are in bijective correspondence with sections of the fibre bundle $t \colon E/H \to X$. Equivalently, such reductions are in bijective correspondence with G-equivariant continuous maps from E to G/H.*

Proof. We will prove in the next lecture that sections of $E/H \xrightarrow{t} X$ are in bijective correspondence with G-equivariant maps from E to G/H. In fact we will prove, more generally, that sections of $\pi_F \colon E(F) \to X$ are in one-to-one correspondence with G-equivariant maps $\varphi \colon E \to F$.

Next, we have a commutative diagram which is formally like (3.1.1) namely

(3.2.3)



If we have a section $\sigma \colon X \to E/H$ of $t \colon E/H \to X$, then $\sigma^*(3.2.3)$ is a diagram of the form (3.1.1) with $P = \sigma^*E$, $i = \sigma^*\widetilde{\sigma}_0$, etc. Thus we have a reduction of the structure group for E to H.

Conversely, suppose we have a reduction of structure group of E to H. In other words we have a commutative diagram as in (3.1.1). Since $p \colon P \to X$ is a principal H-bundle, we have local sections $s_\alpha \colon U_\alpha \to p^{-1}(U_\alpha)$ of p such that $\mathscr{U} = \{U_\alpha\}$ forms an open cover of X. Let $(h_{\alpha\beta})$ be the corresponding 1-cocycle for the local trivialisations of $p\colon P \to X$ given by the s_α. Recall that for a pair of indices (α, β), $h_{\alpha\beta}\colon U_{\alpha\beta} \to H$ is the unique map satisfying

$$s_\beta(u) = s_\alpha(u)h_{\alpha\beta}(u) \qquad (u \in U_{\alpha\beta}).$$

Let $\tau_\alpha = i \circ s_\alpha$. Then τ_α is a local section of $E \to X$ over U_α. Moreover, since i is H-equivariant, we have

$$\tau_\beta(u) = \tau_\alpha(u)h_{\alpha\beta}(u) \qquad (u \in U_{\alpha\beta}).$$

Let $\sigma_\alpha = \varpi \circ \tau_\alpha\colon U_\alpha \to E/H$. Since $\varpi(eh) = \varpi(e)$ for all $e \in E$ and $h \in H$, it follows that

$$\sigma_{\alpha}|_{U_{\alpha\beta}} = \sigma_{\beta}|_{U_{\alpha\beta}}$$

for all α and β. Thus we get a section

$$\sigma\colon X \to E/H$$

of $t\colon E/H \to X$. □

Remark 3.2.4. If $\varphi\colon E \to G/H$ is G-equivariant, then the corresponding H-subbundle of $\pi\colon E \to X$ can be obtained directly by setting $P = \varphi^{-1}(\xi_0)$ where ξ_0 is the distinguished point of G/H, namely the image of the identity element $\varepsilon \in G$ under the natural map $G \to G/H$. Since φ is G-equivariant, and since ξ_0 is H-invariant, clearly P is H-stable. It is not hard to see that $P \to X$ is in fact a principal H bundle and the inclusion of P into E is H-equivariant.

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