LECTURE 16

1. **Equivalence relations**

1.1. **Equivalence relations and co-equalizers.** The notion of an equivalence relation on a set has the following natural generalization in the category $\mathbb{S}\mathrm{ch}_{/S}$.

Definition 1.1.1. Let $X \in \mathbb{S}\mathrm{ch}_{/S}$. A *schematic equivalence relation* on X over S is an object $R \in \mathbb{S}\mathrm{ch}_{/S}$ together with a morphism $f\colon R \to X \times_S X$ such that for every $T \in \mathbb{S}\mathrm{ch}_{/S}$ the map of sets

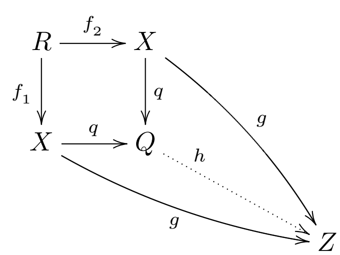
$$f(T)\colon R(T) \to X(T) \times X(T)$$

is *injective* and its image is (the graph of) an equivalence relation on the set $X(T)$. Here, for any $Z \in \mathbb{S}\mathrm{ch}_{/S}$, in keeping with our identification of Z with the functor h_Z, the set $Z(T)$ denotes the set $h_Z(T) := \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T, Z)$ for any $T \in \mathbb{S}\mathrm{ch}_{/S}$.

For example, the scheme T'' in (1.2.1) is a schematic equivalence relation on T' over S, or more precisely, the natural map $T'' \to T \times_S T$, is a schematic equivalence relation on T' over S. We will see—from the definition we give below of quotients by equivalence relations—that $p\colon T' \to T$ is the scheme theoretic quotient of T' with respect to this equivalence relation.

Definition 1.1.2. Let $f\colon R\colon X$ be an equivalence relation on $X \in \mathbb{S}\mathrm{ch}_{/S}$ and $f_1, f_2\colon R \rightrightarrows X$ the natural maps arising from f and the projections $X \times_S X \rightrightarrows X$. A morphism $q\colon X \to Q$ in $\mathbb{S}\mathrm{ch}_{/S}$ s a *quotient* for $R \to X$ (or simply of X by R) if $q \circ f_1 = q \circ f_2$ and given any map $g\colon X \to Z$ in $\mathbb{S}\mathrm{ch}_{/S}$ satisfying $g \circ f_1 = g \circ f_2$ there is a unique map $h\colon Q \to Z$ such that $g = h \circ q$, in other words, as in (1.2.1), if—in the diagram below—the solid arrows form a commutative diagram, then the dotted arrow can be filled in a unique way to make the whole diagram commute:

(1.1.2.1)



If the quotient $q\colon X \to Q$ of X by R exists, then we say it is an *effective quotient* if the natural map $(f_1, f_2)\colon R \to X \times_Q R$ is an isomorphism, i.e., if the square in Diagram (1.1.2.1) is cartesian. We often denote the quotient Q, if it exists, by X/R.

Remark 1.1.3. Clearly, from the universal property of quotients by (schematic) equivalence relations, if such a quotient $q\colon X \to Q$ exists, it is unique up to unique isomorphism. In category theory terms, the universal property of $q\colon X \to Q$ makes

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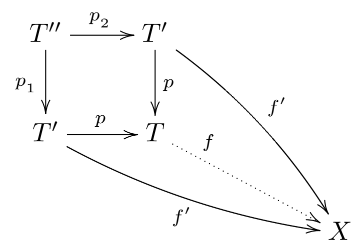
it a *co-equalizer* for the maps f_1 and f_2. Co-equalizers are clearly unique up to unique isomorphisms.

1.2. **Example.** We have seen that every S-scheme X is an fpqc-sheaf on $\mathbb{S}\mathrm{ch}_{/S}$. Recall that this means the following: Suppose $p\colon T' \to T$ is an fpqc-map and as usual we set $T'' := T' \times_S T'$, and let $p_1, p_2\colon T'' \rightrightarrows T'$ denote the two projections. Suppose we have a map $f'\colon T' \to X$ in $\mathbb{S}\mathrm{ch}_{/S}$ such that $f' \circ p_1 = f' \circ p_2$. Then there is a unique map $f\colon T \to X$ such that $f' = f \circ p$. In other words if we have a commutative diagram below of solid arrows in $\mathbb{S}\mathrm{ch}_{/S}$ (with the square being cartesian) then the dotted arrow can be filled in a unique way to make the whole diagram commutative.

(1.2.1)



Here our attention is on X and the cartesian diagram of T's is allowed to vary. If we transfer our attention to the commutative square (fixing it) and allow X to vary in $\mathbb{S}\mathrm{ch}_{/S}$ then clearly $p\colon T' \to T$ is the quotient of T' by the scheme theoretic equivalence relation T''. Moreover, since the square in Definition 1.2.1 is cartesian, T is an effective quotient. We may write $T = T'/T''$.

In a rough way, one might say that the equivalence relation on T' represented by the scheme T'' is the relation that two points of T' are related if and only if they lie on the same fibre of p, i.e., if and only if their image in T is the same. If we decide to denote the equivalence relation by $\sim$ rather than T'', then in terms of valued points, say $t'_1, t'_2 \in T'(W)$, we have $t_1 \sim t_2$ if and only if $p(t_1) = p(t_2)$, where of course, as is standard in this course $p(t'_i) = p \circ t'_i$, $i = 1, 2$. This last description is rigorous as a little thought will show, since it characterizes T''.

We often use the looser notation $x_1 \sim x_2$ to indicate a scheme-theoretic equivalence relation on a scheme, rather than a scheme R and a map $R \to X$. Here of course, x_1 and x_2 are valued points of X. In other words $x_1 \sim x_2$ is really a short hand for a family of set-theoretic equivalence relations, one on each $X(T)$ as T varies over $\mathbb{S}\mathrm{ch}_{/S}$, in such a way that they are compatible with pull backs via maps $T' \to T$. Thus the symbol $x_1 \sim x_2$ will be a short hand for a family of subsets $R(T)$ of $X(T) \times X(T)$, such that R is functorial. Usually the functorial property will be evident. The underlying assumption will be that R is a representable functor. Thus is $E \to X$ is a G torsor and $F \to S$ a G-locally quasi-affine space over S, then on $E \times_S F$ one sometimes writes $(eg, f) \sim (e, gf)$ to indicate a scheme theoretic equivalence relation on $E \times_S F$, with e, g, and f valued points of E, G and F respectively with the same source. It is clear that this defines a functor R on $\mathbb{S}\mathrm{ch}_{/S}$. It is however not clear that it is representable, or that the quotient exists. It turns out that indeed R is representable and the quotient $E \times_S F/\sim$ exists. In fact the quotient is $E(F)$ as we will see in the next lecture.

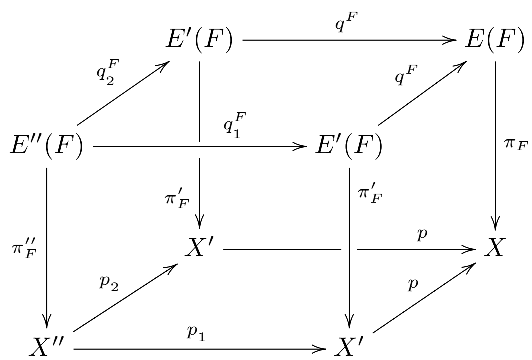
2. Realisation of $E(F)$ as a quotient modulo an equivalence relation

2.1. **Cartesian cube.** Suppose $p\colon X' \to X$ is an fpqc map such that $\pi'\colon E' \to X'$ is trivial. Let

$$\theta\colon G_{X'} \xrightarrow{\ \sim\ } E'$$

be the trivialisation. Then, clearly, we have the identifications $E'(F) = p^*E(F)$, and $E''(F) = p_1^*E'(F) = p_2^*E'(F)$. In fact we have a commutative cartesian cube

(2.1.1)



with q^F and q_i^F being the natural projections. From this the following result can essentially be read off:

Proposition 2.1.2. *The maps* $q^F\colon E'(F) \to E(F)$ *and* $q_i^F\colon E''(F) \to E'(F)$, $i = 1, 2$ *are fpqc maps.*

Proof. The maps $p\colon X' \to X$ and $p_i\colon X'' \to X$ are fpqc. The result follows, since (2.1.1) is a cartesian cube. $\square$

The top face of the cube is:

$$\begin{array}{ccc} E''(F) & \xrightarrow{q_2^F} & E'(F) \\ {\scriptstyle q_1^F}\downarrow & \square & \downarrow{\scriptstyle q^F} \\ E'(F) & \xrightarrow[q^F]{} & E(F) \end{array}$$

Now the trivialisation $\theta\colon G_{X'} \xrightarrow{\ \sim\ } E'$ gives us the isomorphism

$$\theta_F\colon F_{X'} \xrightarrow{\ \sim\ } E'(F).$$

Let

$$r_i = \theta_F^{-1} \circ q_i^F \qquad (i = 1, 2).$$

We then have a cartesian square

(2.1.3)
$$\begin{array}{ccc} E''(F) & \xrightarrow{r_2} & F_{X'} \\ {\scriptstyle r_1}\downarrow & \square & \downarrow \\ F_{X'} & \longrightarrow & E(F) \end{array}$$

Since all arrows are fpqc, and the square is cartesian, from our earlier observations, $E(F)$ is the quotient of $F_{X'}$ by the scheme-theoretic equivalence relation $E''(F)$. In fact $E(F)$ is an effective quotient. One can bring the equivalence relation $E''(F)$

down to set theoretic terms in a very understandable way, namely using $\sim$ as a short hand for the equivalence relation. We have:

$$(x'_1,\, f_1) \sim (x'_2,\, f_2) \Longleftrightarrow p(x'_1) = p(x'_2) \text{ and } f_1 = g_\theta(x'_1,\, x'_2) f_2.$$

We will show this in the next lecture. Moreover, if $X' \to X$ is the map $\pi\colon E \to X$, the above will imply that $E(F)$ is the quotient of $E \times_S F$ by the equivalence relation given by

$$(eg,\, f) \sim (e,\, gf)$$

for $e \in E(T)$, $g \in G(T)$, $f \in F(T)$, and $T \in \mathbb{S}\mathrm{ch}_{/S}$. This is in line with the classical construction of the associated fibre space from a principal bundle.

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