LECTURE 15

1. **Preliminaries**

By a *G-space* F we will mean an S-scheme F on which G acts from the left. We decided to use the term *G-space* rather than G-scheme, for the latter can be confused with an object of $\mathbb{S}\mathrm{ch}_{/G}$. A *locally quasi-affine G-space* F is a G-space such that F can be covered by *G-stable* open subschemes U such that each such U is quasi-affine over S. This means that U is G-stable and if $\phi\colon U \to S$ is the structure map then the natural map of S-schemes $U \to \mathbf{Spec}(\phi_*\mathscr{O}_U)$ is an open immersion. Clearly G-spaces form a category with morphisms being G-equivariant maps, and locally quasi-affine G-spaces form a full subcategory. One of the motivations for this course is to construct the fibre space $E(F) \to X$ associated with a G-torsor $E \to X$. The construction is carried out below (the definition and characterizing property are given in (2.2.4)). Our construction uses arbitrary trivialisations of the torsor E. The standard construction of $E(F)$ uses $E \to X$ as the fpqc trivializing cover of E, and we deal with this in later lectures.

2. **Construction of the associated fibre space**

2.1. For any $T \in \mathbb{S}\mathrm{ch}_{/S}$, F_T will denote the product scheme $T \times_S F$. Note that $F_T \to T$ is a locally quasi-affine G_T-space.

We have a natural isomorphism

$$\psi_{\theta} = \psi_{\theta}(F)\colon p_2^*F_{X'} \xrightarrow{\sim} p_1^*F_{X'} \tag{2.1.1}$$

given by

$$(x'',\, f) \mapsto (x'', g_\theta(x'')f) \tag{2.1.2}$$

for valued-points x'' and f of X'' and F respectively. Formula (**??**) immediately gives the formula

$$p_{12}^*(\psi_\theta) \circ p_{23}^*(\psi_\theta) = p_{13}^*(\psi_\theta). \tag{2.1.3}$$

Thus $(F_{X'},\, \psi_\theta)$ is a descent data on $X' \xrightarrow{p} X$. Since $F \to S$ is a locally quasi-affine G-space, so is $X' \times_S F \to X'$ over X'. Moreover p is fpqc. Standard descent theory shows that $F_{X'}$ "descends" to X. In greater precision, there exists (up to unique isomorphism) a unique X-scheme $E(F)$

$$\pi_F\colon E(F) \to X$$

such that $F_{X'}$ is canonically isomorphic to $p^*E(F)$ and such that the identity $p_2^*p^*E(F) = p_1^*p^*E(F)$ corresponds to the isomorphism $\psi_\theta\colon p_2^*F_{X'} \xrightarrow{\sim} p_1^*F_{X'}$.

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2.2. Independence of $E(F)$ from trivialisations. The above construction depends a-priori on the choice of the trivialzing fpqc-cover $p\colon X' \to X$ as well as the trivialisation $\theta\colon G_{X'} \xrightarrow{\sim} E'$. For precision, let us write $E(F, X', \theta)$ for the scheme $E(F)$ just constructed. We will show natural isomorphisms

$$\mu_{12}\colon E(F, X'_2, \theta_2) \xrightarrow{\sim} E(F, X'_1, \theta_1) \tag{2.2.1}$$

for trvializing data $(X'_i \to X', \theta_i)$, $i = 1, 2$, such that for a third such data (X'_3, θ_3) we have compatibility relations

$$\mu_{12} \circ \mu_{23} = \mu_{13}. \tag{2.2.2}$$

First note that if $v\colon X'_* \to X'$ is an fpqc-map, and $\theta_* := v^*\theta$, the trivialisation induced by θ, then $v^*g_\theta = g_{\theta_*}$. As a consequence $v^*\psi_\theta = \psi_{\theta_*}$. Thus, if the map $\alpha\colon F_{X'} \to E(F, X, \theta)$ is the natural descent map for the data ψ_θ, the composite

$$F_{X'_*} \xrightarrow{\text{via } v} F_{X'} \xrightarrow{\alpha} E(F, X', \theta)$$

is the natural descent map for the data ψ_{θ_*}. It follows that we have the identity

$$E(F, X', \theta) = E(F, X'_*, \theta_*). \tag{2.2.3}$$

In classical terms, given glueing data on an open cover, it and the induced data on a refinement yield the same glued object.

Next suppose $p\colon X' \to X$ is as in the previous subsections. Suppose θ_1 and θ_1 are two trivialisations of $E' \to X'$. The two trivialisations differ by right multiplication by an element $G(X')$. More precisely there an element $c_{12} \in G(X')$ such that if $A_{12}\colon G_{X'} \to G_{X'}$ is given by left multiplication by c_{12}, then $\theta_2 = \theta_1 \circ A_{12}$. Let

$$A^F_{12}\colon F_{X'} \xrightarrow{\sim} F_{X'}$$

be the isomorphism induced by $(x',\, f) \mapsto (x',\, c_{12}(x')f)$. One checks easily that the diagram below commutes:

$$\begin{array}{ccc} p_2^*F_{X'} & \xrightarrow[\sim]{\psi_{\theta_2}} & p_1^*F_{X'} \\ {\scriptstyle p_2^*A^F_{12}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle p_1^*A^F_{12}} \\ p_2^*F_{X'} & \xrightarrow[\psi_{\theta_1}]{\sim} & p_1^*F_{X'} \end{array}$$

Thus A^F_{12} is an isomorphism of descent data. This defines the isomorphism μ_{12} in (2.2.1) when $X'_1 = X'_2 = X'$. Moreover, if $\theta_3\colon G_{X'} \to E'$ is a third trivialisation of E', then it is easy to see that $c_{12}c_{23} = c_{13}$, and this in turn ensures that that (2.2.2) is satisfied. Suppose $v\colon X'_* \to X'$ is an fpqc refinement of the E-trivializing fpqc-cover $p\colon X' \to X$. From the above argument we have an isomorphism $\mu^*_{12}\colon E(F, X'_*, v^*\theta_2) \xrightarrow{\sim} E(F, X'_*, v^*\theta_1)$. One checks that $v^*c_{12} := c_{12} \circ v$ is the element of $G(X'_*)$ which effects the transition from $v^*\theta_2$ to $v^*\theta_1$. Thus, under the identification (2.2.3) above, we have $\mu^*_{12} = \mu_{12}$.

Finally, given any finite set of fpqc-refinements $v_i\colon X'_i \to X'$ of X'—in other words given a finite set of fpqc-maps v_i—we can find a fpqc-refinement $v\colon T' \to X'$ such that T' refines each of the X'_i's. Indeed the fibre-product of the X'_i over X'

would be such a refinement. Using this and what we've shown above, one can find μ_{ij} as in (2.2.1) which satisfy (2.2.2). We leave the details to the reader.

It is clear that $F \rightsquigarrow E(F, X', \theta)$ is functorial in F, i.e., it varies well with G-maps $F \to F'$. Indeed the construction is driven by the "transition function" $g_\theta \colon X'' \to G$ for all fibres, and hence by the master co-cycle relations (**??**). It is easy to see that if $F \to F'$ is a map of locally quasi-affine G-spaces, the map $E(F, X', \theta) \to E(F', X', \theta)$ is compatible with the change of trivialzing data (X', θ) since the maps A^F_{12} and $A^{F'}_{12}$ are driven by the same element $c_{12} \in G(X')$, and we can always assume that the trivialisations θ_1 and θ_2 occur over the same scheme X'. Thus $F \rightsquigarrow E(F)$ is functorial in F. Note that $E(F) \in \mathbb{S}\mathrm{ch}_{/X}$. Let $\pi_F \colon E(F) \to X$ be the structure map.

We are in a position to make a definition:

Definition 2.2.4. Let $\pi \colon E \to X$ be a G-torsor, F a locally quasi-affine G-space. The X-scheme $\pi_F \colon E(F) \to X$ constructed above is called the *fibre space associated to F*. It is characterized by the following property: Let $p \colon X' \to X$ be an fpqc map such that $E' := E_{X'}$ is a trivial G-torsor, and $\theta \colon G_{X'} \xrightarrow{\sim} E'$ a trivialisation of E'. Then there is an isomorphism of $G_{X'}$-spaces

$$\theta_F \colon F_{X'} \xrightarrow{\sim} p^*E(F)$$

such that the $G_{X''}$-space automorphism $p_1^*\theta_F^{-1} \circ p_2^*\theta_F$ of $F_{X''}$ $(= p_i^*F_{X'})$ is given by

$$(x'',\, f) \mapsto (x'',\, g_\theta(x'')f).$$

Here g_θ is the element in $G(X'')$ defined in (**??**) or equivalently in (**??**). In other words, g_θ is the element such that the automorphism $p_1^*\theta^{-1} \circ p_2^*\theta$ of the trivial G-torsor $G_{X''}$ is described by $(x'',\, g) \mapsto (x'',\, g_\theta(x'')g)$.

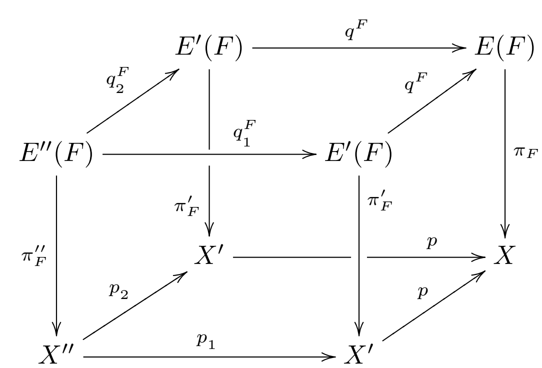
From the argument given just above Definition 2.2.4 we have:

Proposition 2.2.5. *The association $F \rightsquigarrow E(F)$ is a functor from the category of locally quasi-affine G-spaces to the category of schemes over X.*

Remark 2.2.6. If $E \to X$ is a trivial G-torsor, then clearly so is $E(F)$. Indeed, in this case one may take $X' = X$ and the matter is then clear.

2.3. Cartesian cube. Suppose $p \colon X' \to X$ is an fpqc map such that $\pi' \colon E' \to X'$ is trivial. Then, clearly, we have the identifications $E'(F) = p^*E(F)$, and $E''(F) = p_1^*E'(F) = p_2^*E'(F)$. In fact we have a commutative cartesian cube, analogous to—and arising from—diagram (**??**), namely

(2.3.1)



with q^F and q_i^F being the natural projections. From this the following result can essentially be read off:

Proposition 2.3.2. *The maps* $q^F\colon E'(F) \to E(F)$ *and* $q_i^F\colon E''(F) \to E'(F)$, $i = 1, 2$ *are fpqc maps.*

Proof. The maps $p\colon X' \to X$ and $p_i\colon X'' \to X$ are fpqc. The result follows, since (2.3.1) is a cartesian cube. □

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