LECTURE 14

1. Characterisation of torsors

1.1. Facts from Lecture 13. Let $f\colon Z \to X$ be a G-equivariant map with G-acting trivially on X. In our last lecture we showed that the following are equivalent:

- $Z \to X$ is a G-torsor.
- The map f is faithfully flat and $f_Z\colon Z_Z = Z \times_X Z \to Z$ is a trivial G-torsor.
- There exists a smooth surjective map $T \to X$ such that f_T is a trivial G-torsor.
- There exists an étale surjective map $U \to X$ such that $f_U\colon Z_U \to U$ is a trivial G-torsor.

1.2. From the above following is immediate

Corollary 1.2.1. *Let $f\colon Z \to X$ be a G-equivariant map with G-acting trivially on X. The following are equivalent:*

(a) *$Z \to X$ is a G-torsor.*
(b) *There exists an fpqc-map $T \to X$ such that $f_T\colon Z_T \to T$ is a G-torsor.*
(c) *The map f is faithfully flat and $f_Z\colon Z_Z = Z \times_X Z \to Z$ is a G-torsor.*
(d) *There exists a smooth surjective map $T \to X$ such that f_T is a G-torsor.*
(e) *There exists an étale surjective map $U \to X$ such that $f_U\colon Z_U \to U$ is a G-torsor.*

Proof. The conditions (a) and (b) are equivalent because the composites of fpqc-maps are again fpqc. From the result quoted, clearly (a) implies each of (c), (d) and (e). And since smooth maps (and therefore also étale maps) are fpqc (in fact fppf), and composites of fpqc-maps are fpqc, it is evident that (d) implies (a) as does (e). Thus (a), (b), (d), and (e) are equivalent and (a) implies (c).

Now (c) implies that f is smooth, since it is so after a base change by a faithfully flat map (namely the map f itself). Thus (b) implies (c). $\square$

2. Transition elements and cocycles

The aim of this section is to find a natural 1-cocycle (or transition element) in $G(X'')$ arising from a trivialising fpqc-cover $X' \to X$ and the trivialization of E over X'.

2.1. Trivialisations revisited. We have seen that if $\pi\colon E \to X$ is a G-torsor, and if $e\colon X \to E$ is a section then $\psi_e\colon G_X \to E$ given by $(x, g) \mapsto e(x)g$ is a G-equivariant isomorphism of X-schemes (for the right G-structure on G_X). In other words the section e gives us a trivialization ψ_e. Conversely, it is obvious that given a trivialisation $\gamma\colon G_X \xrightarrow{\sim} E$ of E [1], we have a section $e\colon X \to E$ corresponding to the identity section of the group-scheme $G_X \to X$. It is easy to see that $\gamma = \psi_e$. The two processes are evidently inverses of each other.

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[1]In other words, γ is a G-equivariant isomorphism of X-schemes.

Let us agree to denote the identity element of the group $G(S)$ by ε. Note that $\varepsilon\colon S \to G$ is the identity section of the group scheme $G \to S$. If $T \in \mathbb{S}\mathrm{ch}_{/S}$, let the identity of $G(T)$ be denoted $\varepsilon(T)$. Note that $\varepsilon(T)$ is the composite $T \to S \xrightarrow{\varepsilon} G$. Denote by ε_T the identity element of $G_T(T)$. Clearly ε_T is the identity section of the group-scheme $G_T \to T$. The section ε_T can also be regarded as the graph of the map $\varepsilon(T)\colon T \to G$, i.e.,

$$\varepsilon_T = (1_T,\, \varepsilon(T)).$$

In particular if $t \in T(W)$ and $g \in G(W)$ then

$$(t,\, g) = \varepsilon_T(t)g. \tag{2.1.1}$$

The situation is summarized by the commutative square below (with the curves arrows being sections):

$$\begin{tikzcd} G_T \arrow[r] \arrow[d] & G \arrow[d] \\ T \arrow[u, bend left, "\varepsilon_T"] \arrow[r] \arrow[ur, "\varepsilon(T)"] & S \arrow[u, bend right, "\varepsilon"'] \end{tikzcd} \tag{2.1.2}$$

Finally note that by (2.1.1), for a section $e\colon X \to E$ of the torsor $\pi\colon E \to X$, the isomorphism $\psi_e\colon G_X \xrightarrow{\sim} E$ has the alternate description:

$$\varepsilon_X(x)g \mapsto e(x)g. \tag{2.1.3}$$

Here, as usual, x is a valued point of X, g a valued point of G having the same source as x.

Lemma 2.1.4. *Let $\pi\colon E \to X$ be a G-torsor, and e_1, e_2 two sections of π. Then there exists a unique element $g_{12} \in G(X)$ such that*

$$e_2 = e_1 g_{12}.$$

Moreover, if $\psi_{12}\colon G_X \xrightarrow{\sim} G_X$ is the G-equivariant isomorphism of X-schemes [2] *given by $\psi_{12} = \psi_{e_1}^{-1} \circ \psi_{e_2}$ then ψ_{12} is described by $(x,\, g) \mapsto (x,\, g_{12}(x)g)$ for valued points x of X and g of G having the same source.*

Proof. For the first part, since sections of π exist, E is trivial, and we may as well assume, without loss of generality that $E = G_X$. Now given an element $g \in G(X)$, its graph $(1_X,\, g)\colon X \to G_X$ gives a section of $G_X \to X$ and all sections of $G_X \to X$ clearly arise this way. Thus $e_i = (1_X,\, g_i)$ for $i = 1, 2$, with g_1 and g_2 elements in the group $G(X)$. Take $g_{12} = g_1^{-1} g_2$. Clearly $e_2 = e_1 g_{12}$. This proves the first part.

For the second part, keeping in mind (2.1.1), we have to show that ψ_{12} is given by

$$\varepsilon_X(x)g \mapsto \varepsilon_X(x)g_{12}(x)g$$

x and g are valued points of X and G respectively having the same source T. Using (2.1.3) repeatedly, and (2.1.1) we have

$$\begin{aligned} \psi_{12}(T)(\varepsilon_X(x)g) &= (\psi_{e_1}^{-1}(T) \circ \psi_{e_2}(T))(\varepsilon_X(x)g) \\ &= \psi_{e_1}^{-1}(T)(e_2(x)g) \\ &= \psi_{e_1}^{-1}(T)(e_1(x)g_{12}(x)g) \\ &= \varepsilon_X(x)g_{12}(x)g, \end{aligned}$$

[2] With respect to the right G-action on G_X.

as required. □

2.2. Cocycles. Let $\pi\colon E \to X$ be a G-torsor. Let $p\colon X' \to X$ be an fpqc-map such that $\pi_{X'}\colon E_{X'} \to X'$ is a trivial G-torsor. Let $X'' = X' \times_X X'$ and $p_1, p_2\colon X'' \rightrightarrows X'$. We have a cartesian square:

(2.2.1)
$$\begin{tikzcd} X'' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "p"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

Set $E' := E_{X'}$ and $E'' := E_{X''}$ and let $\pi'\colon E' \to X'$ and $\pi''\colon E'' \to X''$ be the resulting base changes of π . Let $q\colon E' \to E$ be the map lying over p, and q_1, $q_2\colon E'' \rightrightarrows E'$ the maps lying over p_1, $p_2\colon X'' \rightrightarrows X'$. We have a commutative cartesian cube (all six faces being cartesian) with the bottom square equal to (2.2.1)

(2.2.2)
$$\begin{tikzcd} & E' \arrow[rr, "q"] \arrow[dd, "\pi'"] & & E \arrow[dd, "\pi"] \\ E'' \arrow[ur, "q_2"] \arrow[rr, "q_1"] \arrow[dd, "\pi''"'] & & E' \arrow[ur, "q"] \arrow[dd, "\pi'"] & \\ & X' \arrow[rr, "p"] & & X \\ X'' \arrow[ur, "p_2"] \arrow[rr, "p_1"] & & X' \arrow[ur, "p"] & \end{tikzcd}$$

Note that the diagram summarizes a lot of data, including the identities $E' = p^*E$ and $E'' = p_1^*E' = p_2^*E'$.

Next fix a a G-invariant isomorphism

$$\theta\colon G_{X'} \xrightarrow{\sim} E'.$$

(This is equivalent to fixing a section of $\pi'\colon E' \to X'$.) We have an isomorphism

$$\varphi_\theta\colon p_2^*G_{X'} \xrightarrow{\sim} p_1^*G_{X'} \tag{2.2.3}$$

given by the formula

$$\varphi_\theta = p_1^*(\theta)^{-1} \circ p_2^*(\theta). \tag{2.2.4}$$

We claim φ_θ gives a descent datum. There are two ways of seeing this. The first way is to note that E is in fact the descent of E', for $E' = p^*E$, and φ is the natural descent data that arises from such a descent. The second way is simply an elaboration of what was just said. Indeed, let $X''' = X' \times_X X' \times_X X'$, $E''' = E_{X'''}$, and let p_{12}, p_{23}, and p_{13} be the three projections $X''' \to X''$. The situation is summarized by the commutative cartesian cube:

$$\begin{tikzcd}
 & X'' \arrow[rr, "p_2"] \arrow[dd, "p_1"'] & & X' \arrow[dd, "p"] \\
X''' \arrow[ur, "p_{23}"] \arrow[rr, "p_{13}"] \arrow[dd, "p_{12}"'] & & X'' \arrow[ur, "p_2"] \arrow[dd, "p_1"] & \\
 & X' \arrow[rr, "p"] & & X \\
X'' \arrow[ur, "p_2"] \arrow[rr, "p_1"'] & & X' \arrow[ur, "p"'] &
\end{tikzcd}$$

We have to check that

$$p_{12}^*(\varphi_\theta) \circ p_{23}^*(\varphi_\theta) = p_{13}^*(\varphi_\theta). \tag{2.2.5}$$

This is easily verified by applying the formulas $p_{12}^* p_2^* = p_{23}^* p_1^*$, $p_{12}^* p_1^* = p_{13}^* p_1^*$, and $p_{13}^* p_2^* = p_{23}^* p_2^*$, to the identity (2.2.4).

Note that

$$p_i^* G_{X'} = G_{X''} = X'' \times_S G \qquad (i = 1, 2).$$

For each valued point $x'' \colon T \to X''$ of X'', we have an element $g_\theta(x'')$ of the group $G(T)$ such that $\varphi_\theta \colon X'' \times_S G \to X'' \times_S G$ is given at the level of valued-points by $(x'', g) \mapsto (x'', g_\theta(x'')g)$. The map $X''(T) \to G(T)$ given by $x'' \mapsto g_\theta(x'')$ is functorial in $T \in \mathbb{S}\mathrm{ch}_{/S}$, and hence we have a map of S-schemes:

$$g_\theta \colon X'' \to G. \tag{2.2.6}$$

There is another simpler description of g_θ. The trivialisation θ corresponds to a (unique) section $e' \colon X' \to E'$ of the torsor $\pi' \colon E' \to X'$. The maps p_1 and p_2 induce sections e_1'' and e_2'' of $\pi'' \colon E'' \to X''$, with $e_i'' = (1_{X''}, e'(p_i))$ for $i = 1, 2$. Applying Lemma 2.1.4 we see that $g_\theta \in G(X'')$ is the unique element satisfying the equation $e_2'' = e_1'' g_\theta$. Equivalently, $g_\theta \in G(X'')$ is the unique element satisfying :

$$e'(p_2) = e'(p_1) g_\theta. \tag{2.2.7}$$

For $(i, j) \in \{(1, 2),\, (2, 3),\, (1, 3)\}$ we have maps $g_{ij}^\theta \colon X''' \to G$ given by

$$g_{ij}^\theta := g_\theta \circ p_{ij} = g_\theta(p_{ij}). \tag{2.2.8}$$

Now, the complete description of the map $p_{ij}^*(\varphi_\theta) \colon G_{X'''} \to G_{X'''}$ from the valued-points point of view is clearly:

$$(x''',\, g) \mapsto (x''',\, g_{ij}^\theta(x''')g).$$

An immediate consequence of this description and the cocycle condition (2.2.5) is the cocycle condition for the g_{ij}^θ's, namely:

$$g_{12}^\theta g_{23}^\theta = g_{13}^\theta. \tag{2.2.9}$$

There is another way of describing (2.2.9). We point out that for $(x_1, x_2, x_3) \in X'''(T)$ we have

$$g^{\theta}_{12}(x_1, x_2, x_3) = g_\theta(x_1, x_2),$$
$$g^{\theta}_{23}(x_1, x_2, x_3) = g_\theta(x_2, x_3),$$
$$g^{\theta}_{13}(x_1, x_2, x_3) = g_\theta(x_1, x_3),$$

and hence (2.2.9) amounts to saying

$$g_\theta(x_1, x_3) = g_\theta(x_1, x_2) g_\theta(x_2, x_3) \tag{2.2.10}$$

for all valued points (x_1, x_2, x_3) of X'''. Note that (2.2.7) is equivalent

$$e'(x_2) = e'(x_1) g_\theta(x_1, x_2)$$

for valued points (x_1, x_2) of X''. We thus have

$$\begin{aligned} e'(x_1) g_\theta(x_1, x_3) &= e'(x_3) \\ &= e'(x_2) g_\theta(x_2, x_3) \\ &= e'(x_1) g_\theta(x_1, x_2) g_\theta(x_2, x_3). \end{aligned}$$

Since the action of $G(T)$ on $E'(T)$ is free for every X'-scheme T, we get (2.2.10), whence (2.2.9).

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