LECTURE 13

1. Basic Lemmas

1.1. **Orbit maps and sections.** Fix a G-equivariant map

$$f\colon Z \to X \tag{1.1.1}$$

where Z and X are S-schemes, with G-actions on them, and such that the action on X is trivial. Suppose f has a section $z\colon X \to Z$, i.e., an X-valued point of Z. We then have a G-equivariant map (the orbit map)

$$\psi_z\colon G_X = X \times_S G \to Z \tag{1.1.2}$$

given by $(x, g) \mapsto z(x)g$ for x and g valued points of X and G respectively and $z(x)$ the composite $z \circ x$.

Lemma 1.1.3. *Let $f\colon Z \to X$ be as in* (1.1.1). *The following are equivalent:*

- (a) *$Z \to X$ is a trivial G-torsor.*
- (b) *There exists a section $z\colon X \to Z$ of f such that the resulting orbit map ψ_z is an isomorphism.*
- (c) *The set of sections of f is non-empty, and for every section z of f, the map ψ_z is an isomorphism.*

Proof. (a)⇒(c): Suppose $Z \to X$ is a trivial G-torsor. Without loss of generality we may assume $Z = G_X$ and that f is the projection $p_X\colon G_X \to X$. By Yoneda all we have to show is that for every $T \in \mathbb{S}\mathrm{ch}_{/X}$, the set theoretic map $\psi_z(T)\colon G_X(T) \to G_X(T)$ is bijective. Indeed if $x\colon T \to X$ is an X-scheme, then $\psi_z(T)$ is the map $(x,\, g) \mapsto (x,\, z(x)g)$ and this is clearly bijective since it amounts to the map on the group $G_X(T)$ given by left multiplication by $(x,\, z(x))$. There is a potential source of confusion in reading the proof here and the next comment is to allay that. Since G_X is an X-scheme, as a functor G_X is regarded as a functor on $\mathbb{S}\mathrm{ch}_{/X}$—not on $\mathbb{S}\mathrm{ch}_{/S}$— and T is regarded as an object of $\mathbb{S}\mathrm{ch}_{/X}$ via $T \xrightarrow{x} X$.

The remaining assertions, namely (c)⇒(b) and (b)⇒(a), are obvious. □

1.2. **Torsors.** In this subsection we give various criteria for checking when maps as in (1.1.1) are torsors. We begin with the following basic lemma

Lemma 1.2.1. *Let $\pi\colon E \to X$ be a G-torsor. Then π is affine, fpqc, and smooth.*

Proof. Fix an fpqc-map $u\colon T \to X$ such that $\pi_T\colon E_T \to T$ is a trivial G-torsor. Let $v\colon E_T \to E$ be the base change of u. We therefore have a cartesian square

$$\begin{array}{ccc} E & \xleftarrow{\ v\ } & E_T \\ {\scriptstyle \pi}\downarrow & \square & \downarrow{\scriptstyle \pi_T} \\ X & \xleftarrow[\ u\ ]{} & T \end{array}$$

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We remind the reader that G is an affine S-scheme and this is at the heart of the matter. Since $E_T \cong T \times_S G$ as a T-scheme, and G is affine over S, one sees that π_T is an affine map. But u is faithfully flat. By a problem in the mid-term exams, it follows that π is also affine.

In order to show π is fpqc, it is enough to show that it is faithfully flat, since faithfully flat affine maps are fpqc. Now π is surjective since u and π_T are. In greater detail, $\pi \circ v = u \circ \pi_T$ is surjective, forcing π to be surjective. It remains to show that π is flat. But this too has been done earlier using the fact that u is faithfully flat.

Since π is flat, it is smooth if its fibres are smooth. Now a variety over a field k is smooth if (and only if) its base change over an algebraic closure of k is non-singular. The latter condition is equivalent to saying (by the Jacobian condition) that the base change of the variety to any algebraically closed extension of k is non-singular. Now the geometric fibres of π_T are geometric fibres of π, and π_T is smooth. Moreover u is surjective and we are done. One can also prove the smoothness of π by verifying the infinitesimal lifting property for π. Indeed π_T has the property and then one uses decent for fpqc-maps. $\square$

Remark 1.2.2. It is not hard to see that a torsor is actually a geometric quotient. The definition of such a quotient for S-schemes (rather than varieties over an algebraically closed fields) is given by replacing the points in the classical definitions with valued points, orbits by orbit maps, or more precisely by scheme theoretic images of orbit maps for valued points, and fibres by fibres over valued points (i.e., by fibre products). We leave the details to the reader as an exercise in the functor of points philosophy.

Lemma 1.2.3. *A G-torsor $\pi\colon E \to X$ is trivial if an only if π has a section.*

Proof. The "only if" part is obvious. We have to prove the "if" part. Let $e\colon X \to E$ be a section of π. Let $T \to X$ be an fpqc-map such that $\pi_T\colon E_T \to T$ is trvial, and let $e_T\colon T \to E_T$ be the base change of the section e. According to Lemma 1.1.3 the orbit map $\psi_{e_T}\colon G_T \to E_T$ is an isomorphism. Consider the cartesian diagram with fpqc horizontal maps (induced by $T \to X$):

$$\begin{array}{ccc} G_X & \longleftarrow & G_T \\ {\scriptstyle \psi_e}\downarrow & \square & \downarrow{\scriptstyle \psi_{e_T}} \\ E & \longleftarrow & E_T \end{array}$$

Since the downward arrow on the right is an isomorphism and $E_T \to E$ is faithfully flat and quasi-compact, the downward arrow on the left is also an isomorphism (see your mid-term exam problems). $\square$

Theorem 1.2.4. *Let $f\colon Z \to X$ be a G-equivariant map with G-acting trivially on X. The following are equivalent:*

(a) *$Z \to X$ is a G-torsor.*
(b) *The map f is faithfully flat and $f_Z\colon Z_Z = Z \times_X Z \to Z$ is a trivial G-torsor.*
(c) *There exists a smooth surjective map $T \to X$ such that f_T is a trivial G-torsor.*
(d) *There exists an étale surjective map $T \to X$ such that $f_T\colon Z_T \to T$ is a trivial G-torsor.*

Proof. (a)$\Leftrightarrow$(b): Assume (a). Note that $f_Z\colon Z_Z = Z \times_X Z \to Z$ is a G-torsor since we have shown in the previous lecture that the base change of a G-torsor is again a G-torsor. It has a section, namely the diagonal section $\Delta\colon Z \to Z \times_X Z$. Now use Lemma 1.2.3 to see that (b) is true. For the converse, we assume (b) and we note that f is smooth and affine (hence fpqc) since it is so after a base change by a faithfully flat map (namely f itself). The statement (a) is obvious from the definition of a G-torsor, since $Z \to X$ base changes to a trivial G-torsor when the base is changed to Z via f.

(b)$\Rightarrow$(c): If f is as (b) then we have seen f is a torsor and therefore smooth by Lemma 1.2.1. With T equal to the X-scheme Z, (c) is immediate from (b).

(c)$\Rightarrow$(d): Fix a point of $x \in X$. We can find an étale neighbourhood U_x of X such that the base change $T_{U_x} \to U_x$ of $T \to X$ has a section (see Section **??**). Let $U = \coprod_x U_x$. Then $U \to X$ is étale and surjective. The map $T_U \to U$ has a section $\varphi\colon U \to T_U$. Consider the map

$$U \to T$$

given by the composite

$$U \xrightarrow{\varphi} T_U \to T.$$

Since $f_T\colon Z_T \to T$ is a trivial G-torsor, so is $f_U\colon Z_U \to U$.

(d)$\Rightarrow$(a) Obvious since étale surjective maps are fpqc.

$\square$

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