LECTURE 12

1. Preliminaries

We will work throughout with a base scheme S. The reader is free to regard S as the spectrum of an algebraically closed field k. The category of S-schemes will be denoted $\mathbb{S}\mathrm{ch}_{/S}$. For the rest of this discussion fix a *smooth* S-group-scheme G which is *affine* over S. By a *G-space* F we will mean an S-scheme F on which G acts from the left. We decided to use the term *G-space* rather than G-scheme, for the latter can be confused with an object of $\mathbb{S}\mathrm{ch}_{/G}$. A *locally quasi-affine G-space* F is a G-space such that F can be covered by *G-stable* open subschemes U such that each such U is quasi-affine over S. This means that U is G-stable and if $\phi\colon U \to S$ is the structure map then the natural map of S-schemes $U \to \mathbf{Spec}(\phi_*\mathscr{O}_U)$ is an open immersion. Clearly G-spaces form a category with morphisms being G-equivariant maps, and locally quasi-affine G-spaces form a full subcategory. One of the motivations for these notes is to construct the fibre space $E(F) \to X$ associated with a G-torsor $E \to X$. The construction is carried out in Subsection **??** (the definition and characterizing property are given in (**??**)). Our construction uses arbitrary trivialisations of the torsor E. The standard construction of $E(F)$ uses $E \to X$ as the fpqc trivializing cover of E, and we deal with this in Subsection **??**

2. Definitions

2.1. **Group actions on schemes.** Let $X \in \mathbb{S}\mathrm{ch}_{/S}$. Then $G_X := X \times_S G$ is a smooth X-group-scheme. Note that we have a right action of G on G_X:

$$G_X \times_S G \to G_X \tag{2.1.1}$$

given by

$$((x,\, g),\, g') \mapsto (x,\, gg') \tag{2.1.2}$$

for "points" g, g' of G and x of X all lying over the same point of S. In somewhat greater detail, in (2.1.2), g, g', and x are *valued points*, i.e., $g, g'\colon T \to G$ and $x\colon T \to X$ are S-maps. A standard functor of points argument (Yoneda Lemma) then says that if in (2.1.2), g, g', and x are arbitrary T-valued points over S, for arbitrary $T \in \mathbb{S}\mathrm{ch}_{/S}$, then we have actually defined a map $G_X \times_S G \to G_X$, and this is our definition of (2.1.1)[1].

We can generalize the idea above. Suppose $f\colon Z \to X$ is a map in $\mathbb{S}\mathrm{ch}_{/S}$ and suppose the S-group-scheme G acts on Z (say from the right) in such a manner that f is G-equivariant for the trivial action of G on X. Then for $T \in \mathbb{S}\mathrm{ch}_{/X}$, if we

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[1] We will constantly use the functor of points technique, and will define maps of schemes by their values on arbitrary "points", with the caveat that these are arbitrary valued points, and therefore do indeed define maps of schemes via the Yoneda Lemma.

set $Z_T := T \times_X Z$, then there is a G action on Z_T,

$$Z_T \times_S G \to Z_T \tag{2.1.3}$$

described by

$$((t,\, z), g) \mapsto (t,\, zg) \tag{2.1.4}$$

for t, z, and g points of T, Z, and G respectively. As usual, these are valued points sharing the same source in $\mathbb{S}\mathrm{ch}_{/S}$.

Remark 2.1.5. Note that (2.1.3) is equivalent to giving a G_T-action on the T-scheme Z_T, and the resulting map

$$f_T \colon Z_T \to T$$

is G-equivariant and G_T-equivariant for the trivial G and G_T actions on T. The reader's attention is drawn to the identification $Z_T \times_T G_T = Z_T \times_S G$.

Definition 2.1.6. Let $f \colon Z \to X$ be as above. We say f is a *trivial G-torsor* if there is a G-equivariant isomorphism of X-schemes:

$$G_X \xrightarrow{\sim} Z. \tag{2.1.6.1}$$

We say f is a *G-torsor* if there exists an fpqc-map $T \to X$ such that the induced map $f_T \colon Z_T \to T$ is a trivial G-torsor. If $Z \to X$ is a G-torsor (resp. trivial G-torsor), we often say Z *is a G-torsor (resp. trivial G-torsor) over X*. Suppose Z and Z' are G-torsors G. A morphism of G-torsors over X from Z to Z' is a G-equivariant map of X-schemes $Z \to Z'$. In particular the diagram

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ & \searrow & \downarrow \\ & & X \end{array}$$

commutes.

Lemma 2.1.7. *Let $E \to X$ be a G-torsor and let $W \to X$ be a map in $\mathbb{S}ch_{/S}$. Then the map $E_W := W \times_Z E \to W$ is also a G-torsor.*

Proof. If $E \to X$ is trivial then it is obvious that $E_W \to W$ is trivial. If $T \to X$ is fpqc, then so is the base change map $T_W = T \times_S W \to W$. If $E_T \to T$ is trivial, then by our first observation, so it $(E_T)_{T_W} \to T_W$. The result follows from the identity $(E_T)_{T_W} = (E_W)_{T_W}$.

□

Remarks 2.1.8. 1) Recall that faithfully flat maps of affine schemes are always fpqc since affine maps are always quasi-compact maps. Many theorems valid for faithfully flat algebras $A \to B$ remain true for fpqc maps. This is done by reducing to the affine case. Here is an example of the technique: Let us prove that if $f \colon X \to Y$ is a map of schemes then it is an isomorphism if it so after a base change by an fpqc-map. So suppose first that $u \colon T \to Y$ is fpqc[2], and the base

[2]The map u need not be fpqc in the sense of SGA—the quasi-compactness hypothesis need not hold. We follow Vistoli [FGA-ICTP] and use the term for a larger class of maps, namely those which are locally faithfully flat and quasi-compact, i.e. maps $f \colon X \to Y$ which are faithfully flat and Y can be covered by affine open subschemes V_i with the property that for each i there is a quasi-compact open subset U_i of X with $f(U_i) = V_i$. Some étale surjective maps need not be fpqc in the SGA sense but always are in our sense. For example the disjoint union of an infinite étale cover of Y would be fpqc in our sense but need not in the SGA sense.

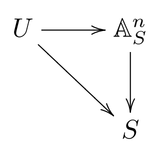
change map $f_T\colon X_T := X \times_Y T \to T$ is an isomorphism. Faithful flatness of u shows that f is affine since f_T is (see the proof of Lemma **??** below for details). So without loss of generality we may assume X and Y are affine and that T is quasi-compact, and hence can be covered by a finite collection of open subsechemes U_α. Let $T' = \coprod_\alpha U_\alpha$. Clearly T' is an affine scheme, for the number of U_α is finite. Moreover the natural map $T' \to T$ is faithfully flat, and hence we have a faithfully flat map $u'\colon T' \to Y$ given by the composite $T' \to T \xrightarrow{f} Y$. Replace u with u' and we are now completely in the affine situation where the statement is well-known.

2) Some statements for faithfully flat maps of rings $A \to B$ carry over to faithfully flat map of schemes $f\colon X \to Y$ without the extra assumption that f is fpqc. Suppose $\mathscr{C}^\bullet$ is a complex of quasi-coherent sheaves on Y such that $f^*\mathscr{C}^\bullet$ is exact. We claim that $\mathscr{C}^\bullet$ is exact. The statement is well known when X and Y are affine. In this instance, by localizing at points of Y, we may assume Y is the spectrum of a local ring A. The closed point of Y has a preimage x in X, since f is faithfully flat and therefore surjective by definition. Let $B = \mathscr{O}_{X,x}$, and we can replace $X \to Y$ by $\operatorname{Spec} B \to Y$ while retaining the hypothesis on the pull-back of $\mathscr{C}^\bullet$. The map $A \to B$ is faithfully flat and we are done.

3) In much the same way as above, one can show that if $f^*\mathscr{F} = 0$ for $\mathscr{F}$ a quasi-coherent sheaf on Y, then $\mathscr{F} = 0$ ($f\colon X \to Y$ faithfully flat). Simpler still, investigate the exactness of the complex $0 \to \mathscr{F} \to 0$ by examining it on X (see (2) above).

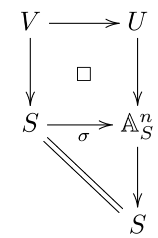
3. **Smooth Maps**

Let us quickly recall some basic facts about smooth maps. Suppose $X \to S$ is smooth of relative dimension n and x is a point on X. We can find an open neighbourhood U of x in X and an étale map $U \to \mathbb{A}^n_S$ such that the diagram



commutes. The algorithm for this (without proofs) is as follows: Without loss of generality we may assume x is closed in its fibre over S, for any open set containing a specialization of x will also contain x. Let $s \in S$ be the image of x and X_s the fibre of $X \to S$ over s. Write $\bar{x} \in X_s$ for the point corresponding to x. Let R be the local ring at x and $\bar{R}$ the local ring at $\bar{x}$. Pick elements $t_1, \dots, t_n \in R$ such that their images in $\bar{R}$ form a regular system of parameters for $\bar{R}$. The elements $t_1, \dots, t_n$ are regular functions (over S) in an open neighbourhood W of x, and so define an S-map $W \to \mathbb{A}^n_S$. This map is étale at x, and so we can find a U as required.

Now suppose $s \in S$. Pick $x \in X$ which is closed in the fibre over s. We wish to show that there is an étale open neighbourhood V of s such that $X \to S$ has a section over V passing through x. More precisely, we can find a section of $X_V := V \times_Y X \to V$ which passes through all points in the inverse image of x in X_V. Clearly $\mathbb{A}^n_S \to S$ is a (smooth) map which has sections (e.g. the so-called *zero section*). Let $\sigma\colon S \to \mathbb{A}^n_S$ be a such a section. Then σ is a closed immersion. Let V be the fibre product of $\sigma\colon S \to \mathbb{A}^n_S$ and $U \to \mathbb{A}^n_S$, and consider the commutative diagram:



It is evident that the map $X_V := V \times_S X \to V$ has a section, for $X_U \to U$ has one (namely, the restriction of the diagonal $X \to X \times_S X$ to U). Clearly, this section passes through all the points in the fibre over x of the map $X_V \to X$.

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