LECTURE 11

1. Right actions and Prinicipal Bundles

1.1. **Problems on Principal Bundles.** In this subsection, we work with the classical notion of a topological space (i.e., not with Grothendieck topologies) and all topological spaces and topological groups that occur are Hausdorff. All group actions of a topological group on a topological space will be assumed to be continuous.

We will deal throughout with a topological group G which acts on the right on a topological space Z, and with a map

$$f\colon Z \to X$$

which is G-equivariant for the trivial action of G on X. We call such an $f\colon Z \to X$ a *G-space over* X, and often simply call Z a G-space over X. Set

$$G_X = X \times G$$

and let

$$\pi_X\colon G_X \to X$$

be the first projection. Note that $g(x, g') = (x, gg')$ gives a left action on G_X and $((x, g')g = (x, g'g)$ a right action on G_X. The space G_X with its right action is clearly G-space over X.

We say $f\colon Z \to X$ is a *trivial* G-space over X if there is a G-equivariant isomorphism (for the right G-action on G_X)

$$\theta\colon G_X \xrightarrow{\sim} Z$$

such that

$$\theta \circ f = \pi_X.$$

Clearly if Z trivial G-space over X then it is a principal bundle over X, in fact the trivial principal bundle.

Proposition 1.1.1. *Let $u\colon W \to X$ be a continuous map and set $Z_W := Z \times_X W$. Let $f_W\colon Z_W \to W$ and $v\colon Z_W \to Z$ be the natural maps. Then G acts naturally on the right on Z_W in such a way that it is a G-space over W and such that v is G-equivariant.*

Proof. Part of your mid-term exam. □

Proposition 1.1.2. *Suppose $\mathscr{U} = \{U_\alpha\}$ is an open cover of X, and Z_{U_α} is a trivial G-space over U_α for each α. Then $f\colon Z \to X$ has the natural structure of a principal G-bundle such that the right G-action on Z induced by the principal bundle is the given G-action on Z.*

Proof. Part of your mid-term exam □

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Proposition 1.1.3. *Suppose the G-action on Z is the one induced by a principal G-bundle structure on $f\colon Z \to X$. Let $u\colon W \to X$ be a continuous map. Then $f_W\colon Z_W \to W$ has a natural structure of a principal G-bundle such that the resulting right G-action on Z_W is the same as the one induced by the right G-action on Z as in Proposition* 1.1.1.

Proof. Part of your mid-term exam $\square$

Proposition 1.1.4. *Consider $\mathscr{Z} := Z_Z = Z \times_X Z$, and the induced map $f_Z\colon \mathscr{Z} \to Z$. Define*

$$\Psi\colon G_Z \to \mathscr{Z}$$

by $(z, g) \mapsto (z, zg)$, $z \in Z$, $g \in G$. Then

(1) *Ψ is G-equivariant for the right G-actions on both spaces.*
(2) *$f_Z \circ \Psi = \pi_Z$.*
(3) *If $f\colon Z \to X$ is a principal G-bundle (such that the induced right G-action is the given one) then Ψ is an isomorphism.*
(4) *Suppose $f\colon Z \to X$ has* local sections, *i.e., around each point $x \in X$ there is an open neighborhood such that the restriction $f^{-1}U_x \to U_x$ of f has a section. Suppose further that Ψ is an isomorphism. Then $f\colon Z \to X$ is a principal bundle and the right G-action on Z induced by its principal bundle structure is the given right G-action on it.*

Proof. Part of your mid-term exam. $\square$

2. The Functor of points

2.1. **Schemes over S as functors.** For any category $\mathscr{C}$, let $\widehat{\mathscr{C}}$ be the category of contravariant (Sets)-valued functors on $\mathscr{C}$. Recall that this means that an object F of $\widehat{\mathscr{C}}$ is a functor

$$F\colon \mathscr{C}^{\circ} \to (\text{Sets})$$

and given two such functors F and G, a morphism from F to G is a natural transformation (or, what is the same thing, a functorial map)

$$F \to G.$$

For the rest of these notes, fix a scheme S, and set $\mathscr{C} := \mathbb{S}\text{ch}_{/S}$. Let X be scheme over S. Define the "functor of points" on X to be the functor on $\mathbb{S}\text{ch}_{/S}$

$$h_X\colon (\mathbb{S}\text{ch}_{/S})^{\circ} \to (\text{Sets})$$

given by

$$T \mapsto \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T, X) \qquad (T \in \mathbb{S}\text{ch}_{/S}),$$

with an obvious effect on morphisms $\varphi\colon T' \to T$ in $\mathbb{S}\text{ch}_{/S}$, namely,

$$q \mapsto q \circ \varphi$$

for $q \in h_X(T) = \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T, X))$. Note that $h_X \in \widehat{\mathscr{C}}$ for every $X \in \mathbb{S}\text{ch}_{/S}$.

Next, if $f\colon X \to Y$ is a map in $\mathbb{S}\text{ch}_{/S}$ then

$$f \circ (\cdot)\colon \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T, X) \to \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T, Y)$$

defined by composing (on the left) with f, is functorial in T. Hence we get a map in $\widehat{\mathscr{C}}$

$$h_f\colon h_X \to h_Y.$$

It is trivial to check that for a pair of maps in $\mathbb{S}\mathrm{ch}_{/S}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

the diagram

$$\begin{array}{ccc} h_X & \xrightarrow{\quad h_f \quad} & h_Y \\ & {\scriptstyle h_{(g\,\circ\, f)}} \searrow \quad \swarrow {\scriptstyle h_g} & \\ & h_Z & \end{array}$$

commutes. In other words the association

$$h_{(\cdot)} \colon \mathbb{S}\mathrm{ch}_{/S} \to \widehat{\mathscr{C}} \tag{2.1.1}$$

defines a functor.

The process $f \mapsto h_f$ (for $f \colon X \to Y$ a map of S-schemes) can be "reversed". More precisely, given a map $\psi \colon h_X \to h_Y$ in $\widehat{\mathscr{C}}$ (X and Y in $\mathbb{S}\mathrm{ch}_{/S}$), we can find a unique map $f = f_\psi \colon X \to Y$ such that $\psi = h_f$. Indeed, we have a map of sets $\psi(X) \colon h_X(X) \to h(Y)$, and hence we have an element $f_\psi \in h_X(Y) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(X, Y)$ defined by the image of $\mathbf{1}_X \in h_X(X) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(X, X)$ under $\psi(X)$. It is easy to see that $h_{f_\psi} = \psi$. It is equally easy to see—from the definitions—that if $f \colon X \to Y$ is a map in $\mathbb{S}\mathrm{ch}_{/S}$ and $\psi \colon h_X \to h_Y$ is defined by $\psi = h_f$, then $f_\psi = f$ (i.e., $f_{h_f} = f$). Thus $f \mapsto h_f$ and $\psi \mapsto f_\psi$ are inverse processes. This can be restated in the following compact form:

$$h_X(T) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T, X) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_T, h_X). \tag{2.1.2}$$

Another way of saying this is that $\mathbb{S}\mathrm{ch}_{/S}$ can be regarded as a full subcategory of $\widehat{\mathscr{C}}$ via the functor $h_{(\cdot)}$ (see Theorem 2.1.4 below).

The isomorphism in (2.1.2) can be extended—as we will see below—to give an isomorphism of sets:

$$F(T) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_T, F). \tag{2.1.3}$$

Indeed, given $\xi \in F(T)$, and $W \in \mathbb{S}\mathrm{ch}_{/S}$, we can define $\theta_\xi(W) \colon h_T(W) \to F(W)$ as follows: Let $f \colon W \to T$ be an element of $h_T(W)$. Writing $f^* = F(f)$, we have $f^* \colon F(T) \to F(W)$. The map $\theta_\xi(W)$ is defined by $f \mapsto f^*(\xi)$. It is easy to see that $\theta_\xi(W)$ is functorial in $W \in \mathbb{S}\mathrm{ch}_{/S}$, whence we have a natural transformation $\theta_\xi \colon h_T \to F$. The association $\xi \mapsto \theta_\xi$ gives us a map $F(T) \to \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_T, F)$. Conversely, given a map $\theta \colon h_T \to F$ in $\widehat{\mathscr{C}}$, we get an element $\xi_\theta \in F(T)$ defined as the image of $\mathbf{1}_T \in h_T(T) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T, T)$ in $F(T)$ under $\theta(T) \colon h_T(T) \to F(T)$. One checks, in the usual way, that $\theta_{\xi_\theta} = \theta$ and $\xi_{\theta_\xi} = \xi$, whence we get the isomorphism (2.1.3).

The isomorphisms (2.1.2) and (2.1.3) are often referred to as the Yoneda lemmas. They are best summarized as a statement, namely:

Theorem 2.1.4. (Yoneda)

(a) *The functor* $h_{(\cdot)} \colon \mathbb{S}ch_{/S} \to \widehat{\mathscr{C}}$ *of* (2.1.1) *is a fully faithful embedding of* $\mathbb{S}ch_{/S}$ *into* $\widehat{\mathscr{C}}$.

(b) *Given* $T \in \mathbb{S}ch_{/S}$ *and* $F \colon (\mathbb{S}ch_{/S})^\circ \to$ (Sets) *a functor, and identifying* T *with* $h_T \in \widehat{\mathscr{C}}$ *via part (a), we have a one-to-one correspondence between* $F(T)$ *and maps* $T \to F$ *in* $\widehat{\mathscr{C}}$.

From now on we will identify $T \in \mathbb{S}\mathrm{ch}_{/S}$ with h_T, and we will treat the isomorphism (2.1.3) as an identity. Thus, with these identifications, we have

$$\mathrm{Hom}_{\widehat{\mathscr{C}}}(T,\, F) = F(T) \qquad (T \in \mathbb{S}\mathrm{ch}_{/S}, F \in \widehat{\mathscr{C}}). \tag{2.1.5}$$

This should be compared with the special case $\mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T,\, X) = X(T)$.

Remark 2.1.6. The alert reader would have recognized that in the proof of Theorem 2.1.4, the category $\mathbb{S}\mathrm{ch}_{/S}$ played no essential role, and could have been replaced by an arbitray category $\mathscr{C}$.

2.2. **The structural morphism for objects in $\widehat{\mathscr{C}}$.** Recall that we are working with schemes over a fixed ambient scheme S. When we write $X \in \mathbb{S}\mathrm{ch}_{/S}$ we are really using a shorthand for $(X \to S) \in \mathbb{S}\mathrm{ch}_{/S}$. The map $X \to S$ is often called the *structural map* or sometimes just the *structure map*. If $S = \operatorname{Spec} A$ is affine, we call $X \in \mathbb{S}\mathrm{ch}_{/S}$ an A-scheme rather than an S-scheme and often write $\mathbb{S}\mathrm{ch}_{/A}$ instead of $\mathbb{S}\mathrm{ch}_{/S}$.

Given an S-scheme X, note that $h_S(X)$ is a singleton set whose only element is the structural map $X \to S$. For $F \in \widehat{\mathscr{C}}$, we have a natural map $F \to h_S$ namely the map such that for $X \in \mathbb{S}\mathrm{ch}_{/S}$, the induced map $F(X) \to h_S(X)$ is the map sending all elements of $F(X)$ to the only element of $h_S(X)$. It is clear that this (as X varies in $\mathbb{S}\mathrm{ch}_{/S}$) is functorial in X. Identifying (as we have agreed to) h_S with S, we thus have a map

$$F \to S \tag{2.2.1}$$

which we call the structural map for F. In the event the object F of $\widehat{\mathscr{C}}$ lies in the smaller category $\mathbb{S}\mathrm{ch}_{/S}$, clearly the above notion of the structural map coincides with the notion defined for schemes over S.

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