LECTURE 10

Our assumptions, conventions and notations are as in Lecture 9. In particular G is a topological group, and all topological spaces that occur (including G) are Hausdorff and all group actions of G on a topological space are continuous unless otherwise stated.

1. Morphisms of fibre bundles with structure groups

1.1. Suppose $\pi\colon \mathscr{F} \to X$ and $\pi'\colon \mathscr{F}' \to X$ are fibre bundles with fibres F and structure group G. A morphism $\mathscr{F} \to \mathscr{F}'$ of fibre bundles with fibre F and structure group G is a continuous map $\phi\colon \mathscr{F} \to \mathscr{F}'$ such that $\pi' \circ \phi = \pi$ satisfying the following requirements: There is a common trivializing cover $\mathscr{U} = \{U_\alpha\}$ for $\mathscr{F} \to X$ and $\mathscr{F}' \to X$, with trivializing maps

$$\theta_\alpha\colon U_\alpha \times F \to \pi^{-1}(U_\alpha)$$

and

$$\theta'_\alpha\colon U_\alpha \times F \to {\pi'}^{-1}(U_\alpha)$$

for each index α such that the restriction

$$\pi^{-1}(U_\alpha) \xrightarrow{\text{via } \phi} {\pi'}^{-1}(U_\alpha)$$

is described as a map

$$U_\alpha \times F \to U_\alpha \times F$$

(using θ_α and θ'_α) by $(u, f) \mapsto (u,\, g_\alpha(u)f)$, where $g_\alpha\colon U_\alpha \to G$ is a continuous map. In other words, we have a collection (g_α) of continuous maps $g_\alpha\colon U_\alpha \to G$ such that for each α, and if $\widetilde{g}_\alpha\colon U_\alpha \times F \to U_\alpha \times F$ is the map $(u,\, f) \mapsto (u,\, g_\alpha(u)f)$ then the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\text{via } \phi} & {\pi'}^{-1}(U_\alpha) \\ {\scriptstyle\theta_\alpha}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle\theta'_\alpha} \\ U_\alpha \times F & \xrightarrow[\widetilde{g}_\alpha]{} & U_\alpha \times F \end{array}$$

commutes for each α. Such morphisms ϕ are actually *gauge morphisms.* They have the property that if $t_{\alpha\beta}\colon U_{\alpha\beta} \to G$ and $s_{\alpha\beta}\colon U_{\alpha\beta} \to G$ are the transition functions for $\mathscr{F}$ and $\mathscr{G}$ with respect to θ and θ', then

$$g_\alpha(u)t_{\alpha\beta}(u) = s_{\alpha\beta}(u)g_\beta(u) \qquad (u \in U_{\alpha\beta})$$

holds for every pair of indices (α, β).

Remark 1.1.1. A morphism of fibre bundles with fibre F and stricture group G is necessarily and isomorphism, for the maps $\widetilde{g}_\alpha$ are clearly homeomorphisms.

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2. Principle bundles through right G-actions

2.1. We have seen that if $\pi\colon P \to X$ is a principal G-bundle then there is a canonical G-action from the right on P such that the map π is equivariant for the trivial right action of G on X. A partial converse is the following theorem

Theorem 2.1.1. *Suppose $\pi\colon \mathscr{F} \to X$ is a continuous map such that there is a right G-action on $\mathscr{F}$ which is equivariant for for the trivial right action on X. The $\pi\colon \mathscr{F} \to X$ is a principal G bundle if and only if there is an open cover $\mathscr{U} = \{U_\alpha\}$ of X together with a G-equivariant homeomorphisms (for the right action of G on $U_\alpha \times G$), one for each index α*

$$\varphi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha)$$

such that the diagram

$$\begin{array}{ccc} U_\alpha \times G & \xrightarrow[\varphi_\alpha]{\sim} & \pi^{-1}U_\alpha \\ & {\scriptstyle projection}\searrow & \downarrow{\scriptstyle via\ \theta} \\ & & U_\alpha \end{array}$$

commutes for every α.

3. 1-Cocycles and the first cohomology set

3.1. The sheaf of G-valued functions. Let X be a topological space. Let $\mathscr{G} = \mathscr{G}_X$ denote the sheaf of G-valued continuous functions on X. Thus

$$\Gamma(U, \mathscr{G}) = \{U \xrightarrow{\alpha} G \mid \alpha \text{ is continuous}\} \qquad (U \text{ open in } X)$$

and the restriction maps are the usual restrictions of functions.

Since G need not be abelian, $\mathscr{G}$ need not be a sheaf of abelian groups on X.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open cover of X. Let

$$C^1(\mathscr{U}, \mathscr{G}) = \prod_{ij\in I\times I} \Gamma(U_{ij}, \mathscr{G}).$$

A typical element of $C^1(\mathscr{U}, \mathscr{G})$ looks like (f_{ij}) where $f_{ij}\colon U \to G$ is a continuous map for each pair of indices (i, j). The element (f_{ij}) is said to be a 1-cocycle if

$$f_{ij}(u)f_{jk}(u) = f_{ik}(u) \qquad \text{for } u \in U_{ijk} \text{ and for every } i, j, k \in I. \tag{3.1.1}$$

Let $Z^1(\mathscr{U}, \mathscr{G})$ denote the set of 1-cocycles in $C^1(\mathscr{U}, \mathscr{G})$. Two 1-cocycles (f_{ij}) and (g_{ij}) are said to be *cohomologous* if there exist $h_i \in \mathscr{G}(U_i)$ such that

$$h_i(u)f_{ij}(u) = g_{ij}(u)h_j(u) \qquad u \in U_{ij}, (i, j) \in I \times I.$$

One checks that this gives an equivalence relation on $Z^1(\mathscr{U}, \mathscr{G})$.

Remark 3.1.2. $Z^1(\mathscr{U}, \mathscr{G})$ has a distinguished element, namely (1_{ij}), where $I_{ij}\colon U_{ij} \to G$ is the map $u \mapsto 1$ where $1 \in G$ is the identity element. Thus $Z^1(\mathscr{U}, \mathscr{G})$ is a pointed set.

Let $\mathrm{H}^1(\mathscr{U}, \mathscr{G})$ be the set of equivalence classes in $Z^1(\mathscr{U}, \mathscr{G})$ with respect to the the equivalence relation "cohomologous to". Note that $\mathrm{H}^1(\mathscr{U}, \mathscr{G})$ has a distinguished element. As in the classical case (of sheaves of *abelian groups*) we define

$$\mathrm{H}^1(\mathscr{U}, \mathscr{G}) \to \mathrm{H}^1(\mathscr{V}, \mathscr{G})$$

if $\mathscr{V}$ is a refinement of $\mathscr{U}$. In greater detail, let $\mathscr{V} = \{V_i\}_{j\in J}$ be a refinement of $\mathscr{U} = \{U_i\}_{i\in I}$, and suppose $\tau\colon J \to I$ is a *refining map*, i.e., $V_j \subset U_{\tau(j)}$ for all $j \in J$. Then a 1-cocycle

$$\sigma = (f_{i_1 i_2}) \in Z^1(\mathscr{U}, \mathscr{G})$$

gives rise to a 1-cocycle $\tau^*\sigma \in Z^1(\mathscr{V}, \mathscr{G})$ given by

$$\tau^*\sigma_{j_1 j_2} = f_{\tau(j_1)\tau(j_2)}|_{V_{j_1 j_2}}.$$

This map $\tau^*\colon Z^1(\mathscr{U}, \mathscr{G}) \to Z^1(\mathscr{V}, \mathscr{G})$ preserves the relation "cohomologous to". We thus get

$$\Phi^{\mathscr{V}}_{\mathscr{U}}\colon \mathrm{H}^1(\mathscr{U}, \mathscr{G}) \to \mathrm{H}^1(\mathscr{V}, \mathscr{G}).$$

Standard arguments show that $\Phi^{\mathscr{V}}_{\mathscr{U}}$ depends only on $\mathscr{U}$ and $\mathscr{V}$ and not on the refining map τ. The map $\Phi^{\mathscr{V}}_{\mathscr{U}}$ is a map of pointed sets. Turns out that

$$\Phi^{\mathscr{W}}_{\mathscr{V}} \circ \Phi^{\mathscr{V}}_{\mathscr{U}} = \Phi^{\mathscr{W}}_{\mathscr{U}}.$$

Define

$$\mathrm{H}^1(X, \mathscr{G}) = \varinjlim_{\mathscr{U}} \mathrm{H}^1(\mathscr{U}, \mathscr{G}).$$

Proposition 3.1.3. *Let X be as above, and let F be a topological space on which G acts from the left. There is a bijective correspondence between isomorphism classes of fibre bundles over X (with structure group G ad fibre F) and* $\mathrm{H}^1(X, \mathscr{G})$.