LECTURE 1

1. Gluing sheaves

Let X be a topological space and $\mathscr{X} = \{X_\alpha \,|\, \alpha \in \Lambda\}$ an open cover of X. For any open subset $U \subset X$, and indices α, β, γ in Λ define $U_\alpha := U \cap X_\alpha$, $U_{\alpha\beta} := U_\alpha \cap U_\beta = U \cap X_{\alpha\beta}$, and $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma = U \cap X_{\alpha\beta\gamma}$. We also will use the notation $\mathscr{X} \cap U$ for the open cover $\{U_\alpha\}$ of U. Suppose that for each $\alpha \in \Lambda$ we have a sheaf $\mathscr{F}_\alpha$ on X_α and isomorphisms

$$\varphi_{\alpha\beta} \colon \mathscr{F}_\beta|_{X_{\alpha\beta}} \xrightarrow{\sim} \mathscr{F}_\alpha|_{X_{\alpha\beta}},$$

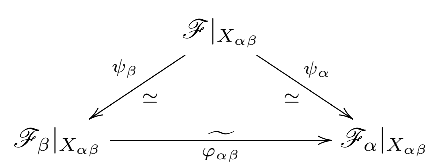
—one for each pair of indices $\alpha, \beta \in \Lambda$—these isomorphisms satisfying the co-cycle rules

$$\varphi_{\alpha\alpha} = 1, \qquad \alpha \in \Lambda$$
$$\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma} \quad \text{on } X_{\alpha\beta\gamma} \quad \alpha, \beta, \gamma \in \Lambda.$$

Then (as we will show) there is a sheaf $\mathscr{F}$ on X together with isomorphisms

$$\psi_\alpha \colon \mathscr{F}|_{X_\alpha} \xrightarrow{\sim} \mathscr{F}_\alpha$$

such that the diagram



commutes. In fact the pair $(\mathscr{F}, \{\psi_\alpha\}_\alpha)$ is unique up to unique isomorphism. Here is how one finds $\mathscr{F}$.

Pick an open set U in X. Define a map $\varphi^*(U) \colon \prod_{\alpha\in\Lambda} \mathscr{F}_\alpha(U_\alpha) \to \prod_{(\alpha,\beta)} \mathscr{F}_\alpha(U_{\alpha\beta})$ by

$$(s_\alpha)_\alpha \mapsto \big(s_\alpha|_{U_{\alpha\beta}} - \varphi_{\alpha\beta}(U_{\alpha\beta})(s_\beta|_{U_{\alpha\beta}})\big)_{\alpha\beta}.$$

Now define

$$\mathscr{F}(U) := \ker(\varphi^*(U)).$$

It is easy to see that the assignment $U \mapsto \mathscr{F}(U)$ is a sheaf.

Where are the co-cycle rules used? In producing the isomorphisms

$$\psi_\alpha \colon \mathscr{F}|_{X_\alpha} \xrightarrow{\sim} \mathscr{F}_\alpha \qquad \alpha \in \Lambda$$

satisfying the requirement that the diagram above commutes for every pair (α, β). Here is a sketch of how that is done. Fix an index $\lambda \in \Lambda$. Let $U \subset X_\lambda$ be an open subset of X_λ. Then, by our notations, $U = U_\lambda$. Pick an element $s \in \mathscr{F}(U)$. Write

$$s = (s_\alpha)_\alpha.$$

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By definition of $\mathscr{F}(U)$ as the kernel of $\varphi^*(U)$, and using the co-cycle rules, we get, for every $\alpha \in \Lambda$,

$$s_\alpha = \varphi_{\alpha\lambda}(s_\lambda|_{U_\alpha}).$$

Thus s_λ determines all the s_α. Note that we are using the fact that U is U_λ, whence $U_{\lambda\alpha} = U_\alpha$. Define

$$\psi_\lambda(U) \colon \mathscr{F}(U) \to \mathscr{F}_\lambda(U)$$

by

$$(s_\alpha)_\alpha \mapsto s_\lambda.$$

One checks, as U varies over open subsets of X_λ, that this defines a map of sheaves $\psi_\lambda \colon \mathscr{F}|_{X_\lambda} \to \mathscr{F}_\lambda$. Since s_λ determines all other s_α in a section $(s_\alpha)_\alpha \in \mathscr{F}(U)$ for $U \subset X_\lambda$, ψ_λ is an isomorphism. There is, as we pointed out in the lecture, another way of doing this. Consider the diagram of exact sequences.

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{F}(U) & \longrightarrow & \prod_{\alpha\in\Lambda} \mathscr{F}_\alpha(U_\alpha) & \xrightarrow{\varphi^*} & \prod_{(\alpha,\beta)} \mathscr{F}_\alpha(U_{\alpha\beta}) \\
 & \downarrow{\scriptstyle \psi_\lambda} & & \downarrow{\scriptstyle \prod_\alpha \varphi_{\lambda\alpha}} & & \downarrow{\scriptstyle \prod_{(\alpha,\beta)} \varphi_{\lambda\alpha}} \\
0 \longrightarrow & \mathscr{F}_\lambda(U) & \longrightarrow & \prod_{\alpha\in\Lambda} \mathscr{F}_\lambda(U_\alpha) & \longrightarrow & \prod_{(\alpha,\beta)} \mathscr{F}_\lambda(U_{\alpha\beta})
\end{array}$$

If we show that it commutes, then ψ_λ has to be an isomorphism, since all other downward arrows are isomorphisms. The only important diagram is the square on the right. The element chase is as follows:

$$\begin{array}{ccc}
(s_\alpha)_\alpha & \longmapsto & (s_\alpha - \varphi_{\alpha\beta}(s_\beta))_{\alpha\beta} \\
\big\downarrow & & \big\downarrow \\
 & & (\varphi_{\lambda\alpha}(s_\alpha) - \varphi_{\lambda\alpha}(\varphi_{\alpha\beta}(s_\beta)))_{\alpha\beta} \\
 & & \big\| \\
(\varphi_{\lambda\alpha}(s_\alpha))_\alpha & \longmapsto & (\varphi_{\lambda\alpha}(s_\alpha) - \varphi_{\lambda\beta}(s_\beta))_{\alpha\beta}
\end{array}$$

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