TORSORS
HOMEWORK 1

Due on Monday August 20, 2012.

Recall that a map of (commutative) rings $A \to B$ is said to be flat if the functor $(\_)\otimes_A B$ (from Mod_A to Mod_B) is exact. We say a map of schemes $f\colon X \to Y$ is *flat* if the functor f^* from quasi-coherent $\mathscr{O}_Y$-modules to quasi-coherent $\mathscr{O}_X$-modules is exact.

(1) Let $f\colon X \to Y$ be a map of schemes. Show that the following are equivalent
  (a) The map of schemes $f\colon X \to Y$ is flat.
  (b) For every $x \in X$, the natural map of rings $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is flat
  (c) For every pair of affine open subschemes $U = \operatorname{Spec} B \subset X$ and $V = \operatorname{Spec} A \subset Y$ such that U maps to V under f, the map of rings $A \to B$ is flat.

(2) Show that flatness is stable under base change. In other words show that if

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

is a cartesian square (i.e., a fibre product diagram), and if f is flat, then so is f'. [*Hint*: Use the previous problem to reduce the problem to the affine case and prove it there.]

(3) Show that the composite of flat maps of schemes is flat.

(4) Let $A \to B$ be a map of rings. Show that the following are equivalent:
  (a) B is flat over A and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.[1]
  (b) A sequence of A-modules

$$(\mathbf{E}) \qquad M' \to M \to M''$$

  is exact if and only if $(\mathbf{E}) \otimes_A B$ is exact.
  (c) A homomorphism of A-modules $M \to M'$ is injective if and only if the associated homomorphism $M' \otimes_A B \to M \otimes_A B$ is injective.
  (d) B is flat over A, and an A-module M is zero if and only if $M \otimes_A B = 0$.
  (e) B is flat over A, and $\mathfrak{m}B \neq B$ for all maximal ideals $\mathfrak{m}$ of A.

Date: August 26, 2012.
[1] As a map of sets.

Definition 0.0.1. A map of rings $A \to B$ is said to be *faithfully flat* if it satisfies any of the equivalent conditions of the previous problem. A map of schemes $X \to Y$ is said to be faithfully flat if it is flat and surjective (as a map of sets).

(5) Let $f\colon X \to Y$ be a map of schemes. Show that the following are equivalent:
  (a) The map f is faithfully flat.
  (b) A sequence $\mathbf{E}$ of quasi-coherent $\mathscr{O}_Y$-modules

$$\mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \tag{$\mathbf{E}$}$$

  is exact if and only if $f^*(\mathbf{E})$ is exact.
  (c) A map $\theta\colon \mathscr{F} \to \mathscr{F}'$ of quasi-coherent $\mathscr{O}_Y$-modules is injective if and only if $f^*\theta$ is injective.
  (d) The map f is flat and $f^*\mathscr{F} = 0$ only if $\mathscr{F} = 0$ for $\mathscr{F}$ a quasi-coherent $\mathscr{O}_Y$-module.

(6) Show the following:
  (a) The composite of faithfully flat maps is faithfully flat.
  (b) Any base change of a faithfully flat map is again faithfully flat. [*Hint*: Suppose we have a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

  with f faithfully flat. From a previous problem we know that f' is flat. One only has to check that it is surjective.]