BERKOVICH SPACES SEMINAR - LECTURE 2

Date of Lecture: January 30, 2020

The infinite interval $[0, \infty)$ will be denoted $\mathbf{R}_+$. It is convenient to include 0 in the set $\mathbf{N}$. Thus, in this seminar, $\mathbf{N} = \{0, 1, 2, \dots\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

All Banach rings will be assumed commutative with 1.

We fix a Banach ring $(\mathscr{A}, \|\ \|)$ throughout this lecture. As in §§2.3 of Lecture 1, its spectrum (or its Berkovich spectrum) $\mathscr{M}(\mathscr{A})$ is the set of bounded seminorms on $\mathscr{A}$ endowed with the weakest topology for which every member of $\{\Psi_f\}_{f\in\mathscr{A}}$ is continuous, where Ψ_f is the map $x \mapsto |f|_x$. Here, as in Lecture 1, $|\ |_x$ is the notation for x when we think of $x \in \mathscr{M}(\mathscr{A})$ as a seminorm rather than as a point.

1. Basics

1.1. **Multiplicative seminorms.** Recall that a seminorm $|\ |$ on $\mathscr{A}$ is said to be *bounded* if there exists a $C > 0$ such that $|f| \le C\|f\|$ for all $f \in \mathscr{A}$. This amounts to saying that the map

$$(\mathscr{A}, \|\ \|) \longrightarrow (\mathscr{A}, |\ |),$$

which the identity map on the underlying ring $\mathscr{A}$, is continuous. If $|\ |$ is *multiplicative* then one can take $C = 1$. Indeed, for each $n \ge 1$ we have $|f|^n = |f^n| \le C\|f^n\| \le C\|f\|^n$, whence $|f| \le \sqrt[n]{C}\|f\|$. Letting $n \to \infty$ the assertion follows.

Lemma 1.1.1. *Suppose $\mathscr{A}$ is a field. A seminorm $|\ |$ on $\mathscr{A}$ is multiplicative if and only if $|f^{-1}| = |f|^{-1}$ for all $f \in \mathscr{A} \smallsetminus \{0\}$.*

Proof. If $|\ |$ is multiplicative clearly $|f^{-1}| = |f|^{-1}$ for all $f \in \mathscr{A} \smallsetminus \{0\}$. For the converse, suppose $|f^{-1}| = |f|^{-1}$ for all $f \in \mathscr{A} \smallsetminus \{0\}$. It is enough to show that for non-zero $f, g \in \mathcal{A}$, $|fg| \ge |f||g|$. Now,

$$|f| = |fgg^{-1}| \le |fg||g^{-1}| = |fg||g|^{-1} \qquad (0 \ne f, g \in \mathscr{A})$$

whence $|f||g| \le |fg|$ as required. $\square$

1.2. **The ring $\mathscr{A}\langle r^{-1}T\rangle$.** The following is an analogue of the ring associated with a rational domain in an affinoid space in rigid analytic geometry.

Definition 1.2.1. For $r > 0$, $\mathscr{A}\langle r^{-1}T\rangle$ is the subring of the power series ring $A[|T|]$ defined by the formula

$$\mathscr{A}\langle r^{-1}T\rangle := \left\{ \sum_{n=0}^{\infty} a_n T^n \in A[|T|] \;\middle|\; \sum_{n=0}^{\infty} \|a_n\| r^n < \infty \right\}.$$

For $\sum_{n=0}^{\infty} a_n T^n \in \mathscr{A}\langle r^{-1}T\rangle$, we set

$$\left\| \sum_{n=0}^{\infty} a_n T^n \right\| = \sum_{n=0}^{\infty} \|a_n\| r^n. \tag{1.2.2}$$

Lemma 1.2.3. *$(\mathscr{A}\langle r^{-1}T\rangle, \| \|)$ is a Banach ring. Moreover, for $a \in \mathscr{A}$, the element $1 - aT \in \mathscr{A}\langle r^{-1}T\rangle$ is a unit if and only if $\sum_n \|a^n\| r^n < \infty$.*

Proof. The first part is left to the reader. As for the second part, note that the formal inverse of $1 - aT$ in $\mathscr{A}[|T|]$ is $\sum_n a^n T^n$ and the latter is a member of $\mathscr{A}\langle r^{-1}T\rangle$ if and only if $\sum_n \|a^n\| r^n < \infty$. □

1.3. The Gel'fand transform and characters.

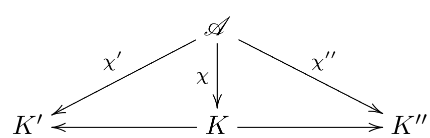
1. Let $x \in \mathscr{M}(\mathscr{A})$.[1] We write $| \,|_x$ for x when we wish to emphasise its role as a multiplicative seminorm. Define the *kernel of* $| \,|_x$ as the set

$$\wp_x = \{f \in \mathscr{A} \mid |f|_x = 0\}.$$

It is straightforward to check that $\wp_x$ is a prime ideal in $\mathscr{A}$. Clearly the residue seminorm of $| \,|_x$ on $\mathscr{A}/\wp_x$ is a muliplicative *norm*, which extends to the quotient field $\mathbf{Q}(x)$ of $\mathscr{A}/\wp_x$ as an absolute value, which we continue to denote $| \,|_x$. Let $\mathscr{K}(x)$ be the completion of $\mathbf{Q}(x)$ with respect to $| \,|_x$. The image of any element $f \in \mathscr{A}$ in $\mathscr{K}(x)$ will be denoted $f(x)$. We have a homomorphism, the so-called *Gel'fand transform*:

$$\widehat{\ }\colon \mathscr{A} \longrightarrow \prod_{x \in \mathscr{M}(\mathscr{A})} \mathscr{K}(x), \qquad f \longmapsto \widehat{f} = (f(x))_{x \in \mathscr{M}(\mathscr{A})}.$$

2. A *character* on $\mathscr{A}$ is a bounded homomorphism $\chi\colon \mathscr{A} \to K$ where K is a field with an absolute value. The kernel is clearly a closed prime ideal of $\mathscr{A}$. Two characters $\chi'\colon \mathscr{A} \to K'$ and $\chi''\colon \mathscr{A} \to K''$ have the same kernel if and only if one can find a character $\chi\colon \mathscr{A} \to K$ (e.g. K the quotient field of $\mathscr{A}/\wp$, where $\wp$ is the common kernel of χ' and χ'') and a commutative diagram



We say χ' and χ'' are equivalent if we can embed them in a commutative diagram as above. It is clear from (i) and the discussion here that $\mathscr{M}(\mathscr{A})$ is in a bijective correspondence with set of equivalence classes of characters. Moreover there is an injective map $\mathscr{M}(\mathscr{A}) \hookrightarrow \operatorname{Spec}_c \mathscr{A}$ where $\operatorname{Spec}_c \mathscr{A}$ is the space of closed prime ideals in $\mathscr{A}$. Indeed if χ_x is a character representing $x \in \mathscr{M}(\mathscr{A})$, then $\wp_x = \ker \chi_x$ a closed prime ideal and the map $\mathscr{M}(\mathscr{A}) \to \operatorname{Spec}_c(\mathscr{A})$, given by $x \mapsto \wp_x$, is one-to-one.

3. Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be a continuous homomorphism of commutative rings. If $| \,|\colon \mathscr{B} \to \mathbf{R}_+$ is a bounded multiplicative seminorm, then so is $| \,| \circ \varphi$ giving us a natural map

$$\varphi^*\colon \mathscr{M}(\mathscr{B}) \longrightarrow \mathscr{M}(\mathscr{A}).$$

This map is clearly continuous. In somewhat greater detail, the topology on $\mathscr{M}(\mathscr{A})$ is generated by sets of the form $\Psi_f^{-1}(B)$ where B is an open ball in $\mathbf{R}$ and $f \in \mathscr{A}$. Here $\Psi_f\colon \mathscr{M}(\mathscr{A}) \to \mathbf{R}$ is the map $x \mapsto |f|_x$. A similar comment is

[1] See §§2.3 of Lecture 1 for a defintion of $\mathscr{M}(\mathscr{A})$.

true for $\mathscr{M}(\mathscr{B})$. Now for $f \in \mathscr{A}$, $(\varphi^*)^{-1}(\Psi_f^{-1}(B)) = \Psi_{\varphi(f)}^{-1}(B)$ showing that φ^* is continuous.

2. Stone-Čech compactifications and products of Banach rings

2.1. **The product of Banach rings.** Suppose $(\mathscr{A}_i)_{i\in I}$ is a family of Banach rings indexed by a nonempty set I. We set $\prod_{i\in I} \mathscr{A}_i$ equal to the set of families $(f_i)_{i\in I}$ with $f_i \in \mathscr{A}_i$ for each $i \in I$ such that the set $\{\|f_i\|,\, i \in I\}$ is bounded in $\mathbf{R}$. Note that this is not the usual definition of a product. One defines a norm on $\prod_i \mathscr{A}_i$ in the obvious way, namely

$$\|(f_i)_{i\in I}\| := \sup_{i\in I} \|f_i\|.$$

Lemma 2.1.1. *$(\prod_{i\in I} \mathscr{A}_i, \|\ \|)$ is a Banach ring.*

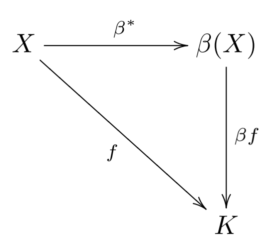
Proof. It is is easy to see that $(\prod_i \mathscr{A}_i, \|\ \|)$ is a normed ring. Since there were some doubts expressed during the lecture about $\prod_i \mathscr{A}_i$ being Banach, here is a proof. If $\{f^{(n)}\}_n = \{(f_i^{(n)})\}_n$ is a Cauchy sequence in $\prod_{i\in I} \mathscr{A}_i$ then each $\{f_i^{(n)}\}$ is clearly a Cauchy sequence in $\mathscr{A}_i$. Let $g_i = \lim_{n\to\infty} f_i^{(n)}$ and $g = (g_i)$. Since $\{(f_i^{(n)})\}_n$ is Cauchy, it is bounded, i.e. there exists a positive real number M such that $\|(f_i^{(n)})\| \le M$, or, what amounts to the same thing, $\|f_i^{(n)}\| \le M$, for $n \in \mathbf{N}$ and $i \in I$. Thus $\|g_i\| \le M$ for $i \in I$, and therefore $g \in \prod_i \mathscr{A}_i$. Next, given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $\|f^{(n)} - f^{(m)}\| < \epsilon$ for $n, m \ge N$. This gives $\|f_i^{(n)} - f_i^{(m)}\| < \epsilon$ for $n, m \ge N$, for each $i \in I$, whence, letting $m \to \infty$, we get

$$\|f_i^{(n)} - g_i\| \le \epsilon \quad \text{for} \quad n \ge N \quad \text{and} \quad i \in I.$$

The integer $N \in \mathbf{N}$ does not depend upon $i \in I$ (from the manner it was chosen). Thus $\|f^{(n)} - g\| \le \epsilon$ for $n \ge N$, and we are done. (The point is that the convergence is uniform over I, and what have given is the standard proof for bounded functions on a set with the sup norm being a complete space.) □

Proposition 2.1.2. *Let $(\mathscr{K}_i)_{i\in I}$ be a family of complete valuation fields and $\mathscr{A} = \prod_{i\in I} \mathscr{K}_i$. Then $\mathscr{M}(\mathscr{A})$ is homeomorphic to the Stone-Čech compactification of I, where I is endowed with the discrete topology.*

Proof. We remind the reader that the Stone-Čech compactification of a topological space X is a compact Hausdorff space $\beta(X)$ together with a continuous map $\beta^*\colon X \to \beta(X)$ such that any continuous map $f\colon X \to K$ from X to a compact Hausdorff space K factors uniquely through β^*. In other words we have unique continuous map $\beta f\colon \beta(X) \to K$ such that the diagram



commutes. If X is locally compact and Hausdorff (e.g. I with the discrete topology) then β^* is an open dense embedding, i.e., the map $X \to \beta^*(X)$ induced by β^* is a homeomorphism with $\beta^*(X)$ dense in $\beta(X)$, and in this case $\beta(X)$ agrees with our intuition of a compactification. Suppose now that X is locally compact and Hausdorff. By the Heine-Borel theorem and the universal property encapsulated by the above commutative diagram, any bounded function $f\colon X \to \mathbf{R}^n$ extends uniquely to a bounded function $\beta f\colon \beta(X) \to \mathbf{R}^n$ and in this case $\sup_{x\in X}\|f(x)\| = \sup_{z\in\beta(X)}\|\beta f(z)\|$.

We now specialise to I with the discrete topology. The general construction of $\beta(X)$ translates to the following construction of $\beta(I)$. As a set $\beta(I)$ is the set of *ultrafilters* on I, a notion which we now define. A *filter* is a family Φ of nonempty subsets of I, i.e. Φ is a filter if $\Phi \subset \mathscr{P}(I) \smallsetminus \{\varnothing\}$ where $\mathscr{P}(I)$ is the power set of I. There is an obvious partial order on filters, namely the one given by inclusions. Ultrafilters are maximal filters in this partial order. Thus, as a set

$$\beta(I) = \{\Phi \mid \Phi \text{ is an ultrafilter}\}.$$

The topology on $\beta(I)$ is as follows. For each non-empty subset J of I let

$$U_J = \{\Phi \mid \Phi \text{ is an ultrafilter on } I \text{ and } J \in \Phi\}.$$

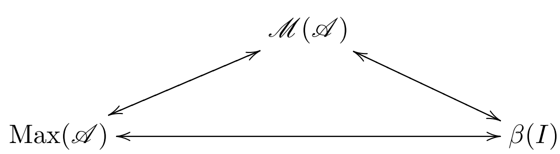
The $\{U_J\}_J$ form a basis for a topology on $\beta(I)$, and this is the topology on $\beta(I)$ giving it the defining universal property. For $i \in I$, if Φ_i is the subset of $\mathscr{P}(I)$ consisting of sets J containing $\{i\}$, then Φ_i is an ultrafilter, and β^* is the map $i \mapsto \Phi_i$.

If $h\colon I \to \mathbf{R}$ is a bounded map, and Φ an ultra filter, then for each $J \in \Phi$ the supremum $\sup_{j\in J} h(j)$ is finite. Moreover $\sup_{j\in J} h(j) \geq \inf_{i\in I} h(i)$ whence

$$(*) \qquad h^*(\Phi) = \inf_{J\in\Phi} \sup_{j\in J} h(j) \in \mathbf{R}.$$

It turns out that $h^*\colon \beta(I) \to \mathbf{R}$ is the unique extension of h to a continuous function on $\beta(I)$. (It is easy to see that $h^*(\Phi_i) = h(i)$ for all $i \in I$.)

For a character $\chi\colon \mathscr{A} \to K$, let $\wp_\chi = \ker \chi$. From **2.** in §§1.3 we see that there is an bijective map of sets from $\mathscr{M}(\mathscr{A})$ to $\{\wp_\chi \mid \chi \text{ a character of } \mathscr{A}\}$. Recall $\wp_\chi$ is a prime ideal. We claim that $\wp_\chi$ is actually a maximal ideal and that every maximal ideal $\mathfrak{m}$ of $\mathscr{A}$ is of the form $\mathfrak{m} = \wp_\chi$ for a character of $\mathscr{A}$. This will give a bijective correspondence between $\operatorname{Max}(\mathscr{A})$ and $\mathscr{M}(\mathscr{A})$, where $\operatorname{Max}(\mathscr{A})$ is the set of maximal ideals of $\mathscr{A}$. We will also establish a bijective correspondence between $\operatorname{Max}(\mathscr{A})$ and $\beta(I)$. In other words, we claim is that there is a commutative diagram of bijective set-theoretic maps



To set up these correspondences we need some notation. If $J \subset I$ let $a_J \in \mathscr{A}$ be the "characteristic function" of $I \smallsetminus J$, i.e. $a_{J,i} = 0$ if $i \in J$, and $a_{J,i} = 1$ if $i \notin J$. For a proper closed ideal $\mathfrak{a}$ of $\mathscr{A}$ let

$$\Phi_\mathfrak{a} = \{J \subset I \mid a_J \in \mathfrak{a}\}.$$

Once checks that $\Phi_\mathfrak{a}$ is a filter on I.[2] Conversely, given a filter Φ on I, let $\mathfrak{a}_\Phi$ be the closed ideal of $\mathscr{A}$ generated by elements of the form a_J for $J \in \Phi$. One checks that

[2] To define $\Phi_\mathfrak{a}$ it is not necessary to assume $\mathfrak{a}$ is closed. However, it is easy to check that $\Phi_\mathfrak{a} = \Phi_{\overline{\mathfrak{a}}}$ where $\overline{\mathfrak{a}}$ is the closure of $\mathfrak{a}$.

$\mathfrak{a} \mapsto \Phi_{\mathfrak{a}}$ and $\Phi \mapsto \mathfrak{a}_{\Phi}$ are inverse maps. Moreover, these correspondences respect the obvious partial orders, whence maximal ideals (which are always closed) are in bijective correspondence with ultra-filters. This gives the horizontal correspondence in the above diagram.

Now for the correspondence indicated on the left (the southwest-northeast correspondence). In view of the conclusions in §§1.3 **2.** it is enough to prove:

(a) Every closed prime ideal is maximal, i.e. $\mathrm{Spec}_c\, \mathscr{A} = \mathrm{Max}(\mathscr{A})$.

(b) If $\mathfrak{m}$ is a maximal ideal on $\mathscr{A}$ then the residue seminorm on $\mathscr{A}/\mathfrak{m}$ is multiplicative.

Let $\wp$ be a closed prime ideal of $\mathscr{A}$ and $\mathfrak{m}$ a maximal ideal such that $\wp \subset \mathfrak{m}$ and $\wp \neq \mathfrak{m}$. Let $J \in \Phi_{\mathfrak{m}} \smallsetminus \Phi_{\wp}$. Then $a_J \in \mathfrak{m} \smallsetminus \wp$. On the other hand $a_{I\smallsetminus J} \notin \mathfrak{m}$, for, if $a_{I\smallsetminus J} \in \mathfrak{m}$, then $1 = a_J + a_{I\smallsetminus J} \in \mathfrak{m}$ contradicting the fact that $\mathfrak{m}$ is a proper ideal. Now $a_J a_{I\smallsetminus J} = 0$, and neither a_J nor $a_{I\smallsetminus J}$ lie in $\wp$. This is a contradiction. This proves (a). For (b), if $\mathfrak{m} \in \mathrm{Max}(\mathscr{A})$, and $\pi\colon \mathscr{A} \to \mathscr{A}/\mathfrak{m}$ is the canonical surjection, then it is easy to see that

$$(**) \qquad \|\pi(f)\|_\pi = \inf_{J\in\Phi_{\mathfrak{m}}} \sup_{j\in J} |f_j|$$

where $\|\ \|_\pi$ is the residue norm on $\mathscr{A}/\mathfrak{m}$. From this it follows that $\|\ \|_\pi$ is multiplicative.

Composing the two bijective correspondences we have established, we get the third one, namely the bijection between $\mathscr{M}(\mathscr{A})$ and $\beta(I)$. We now show that the bijection $\beta(I) \to \mathscr{M}(\mathscr{A})$ so obtained is a homeomorphism.

We regard I in a natural way as a discrete subspace of both $\mathscr{M}(\mathscr{A})$ and $\beta(I)$. If $f = (f_i) \in \mathscr{A}$, then we have a map $f_I\colon I \to \mathbf{R}_+$ given by $i \mapsto |f_i|$. By definition of $\mathscr{A}$, this is a bounded function. The map $\Psi_f\colon \mathscr{M}(\mathscr{A}) \to \mathbf{R}_+$ given by $x \mapsto |f|_x$ is a natural extension of f_I to $\mathscr{M}(\mathscr{A})$. On the other hand, by the universal property of $\beta(I)$, there is a unique continuous bounded map $f_I^*\colon \beta(I) \to \mathbf{R}_+$ extending f_I. From $(*)$ and $(**)$ it is clear that Ψ_f and f_I^* "agree", i.e. are compatible with the bijection between $\mathscr{M}(\mathscr{A})$ and $\beta(I)$. It follows that the bijection $\beta(I) \to \mathscr{M}(\mathscr{A})$ is continuous, since the topology on $\mathscr{M}(\mathscr{A})$ is the weakest topology such that every member the family of functions $\{\Psi_f \mid f \in \mathscr{A}\}$ is continuous. Indeed, the topology on $\mathscr{M}(\mathscr{A})$ is coarser than the one on $\beta(I)$, when we identify the two underlying sets.

It remains to show that $\beta(I) \to \mathscr{M}(\mathscr{A})$ is an open map. Let J be a non-empty subset of I and U_J the corresponding basic open set in $\beta(I)$ defined earlier in this proof. Let V_J be its image in $\mathscr{M}(\mathscr{A})$. Let $f = a_J$. Then f_I^* is zero on U_J and 1 on $\beta(I) \smallsetminus U_J$. It follows that Ψ_f is zero on V_J and 1 on $\mathscr{M}(\mathscr{A}) \smallsetminus V_J$. Since Ψ_f is continuous, we get that V_J is open. □

References

[B] V. G. Bervkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Mathematical Surveys and Monographs, no. **33**, AMS, Providence, Rhode Island, 1990.

[RAG] Course on Rigid Analytic Geometry at CMI. https://www.cmi.ac.in/~pramath/teaching.html#RAG.