BERKOVICH SPACES SEMINAR - LECTURE I

Date of Lecture: January 23, 2020

The infinite interval $[0,\infty)$ will be denoted $\mathbf{R}_+$. It is convenient to include 0 in the set $\mathbf{N}$. Thus, in this seminar, $\mathbf{N} = \{0, 1, 2, \dots\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Summary of Tate algebras and Affinoid algebras

1.1. Absolute values. An absolute value $|\,|$ on a field K is a map

$$|\,|\colon K \to \mathbf{R}_+$$

such that for $x, y \in K$ we have

(a) $|x| = 0 \Leftrightarrow x = 0$,
(b) $|xy| = |x||y|$,
(c) $|x+y| \le |x| + |y|$.

If (c) can be replaced by the stronger condition

(c$'$) $|x+y| \le \max\{|x|,\, |x|\}$,

then $|\,|$ is called *non-archimedean*. It turns out, $|\,|$ is non-archimedean if and only if $|n\cdot 1| \le 1$ for $n \in \mathbf{N}$. The metric corresponding to such an absolute value is an *ultrametric*.

A *valued field* is a pair $(K, |\,|)$ with K a field and $|\,|$ an absolute value on it. There is clearly a notion, via the usual metric space completion via Cauchy sequences, of a completion $\widehat{K}$ of a valued field. This is easily seen to be a field, and $|\,|$ extends to an absolute value on $\widehat{K}$ in a unique way.

On a field K, one always has the trivial absolute value, namely $|x| = 1$ for all $x \neq 0$.

For the rest of this section we will fix a complete non-archimedean valued field $(K, |\,|)$, such that $|\,|$ is non-trivial. Set $\mathscr{O}_K = \{x \in K \mid |x| \le 1\}$ and $\mathfrak{m}_K = \{x \in K \mid |x| < 1\}$. Then $\mathscr{O}_K$ is a valuation ring and $\mathfrak{m}_K$ is its maximal ideal. The field

$$\widetilde{K} := \mathscr{O}_K/\mathfrak{m}_K$$

is called the *residue field*[1] of the valued field K. Since K is complete, every finite extension L of K is complete. In greater detail, there is a unique extension of $|\,|$ to L. Moreover L is complete with respect to this absolute value.[2] It is clear that $|\,|$ extends uniquely to any algebraic closure $\overline{K}$ of K.

[1]The definition of $\mathscr{O}_K$, $\mathfrak{m}_K$ and the notion of a residue field $\widetilde{K}$ do not need the completeness hypothesis, and are in fact important in the non-complete case too.

[2]In general, when K is possibly not complete, if L is a finite extension, the number of extensions of $|\,|$ to L equals the number of maximal ideals in the semi-local artin ring $L \otimes_K \widehat{K}$.

1.2. **Visualising ultrametrics.** There are ways of visualising ultrametrics via trees. Ramadas pointed me towards Jan Holly's AMM article [H], which you can access by clicking here. The article is highly recommended. Here is a sample picture from it.

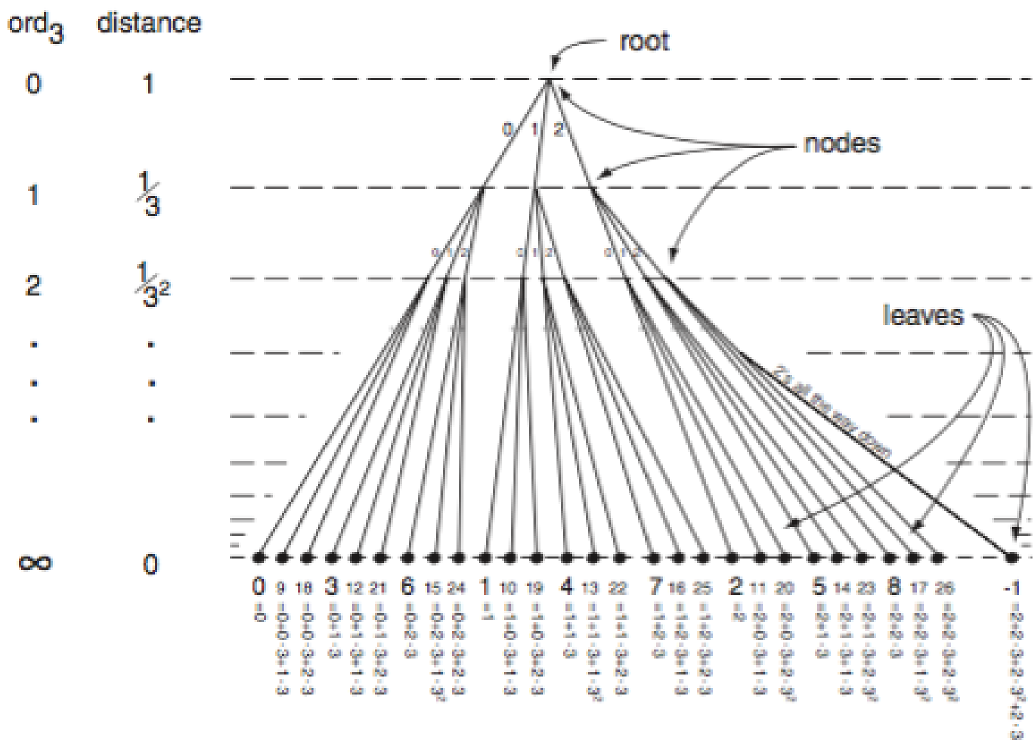

FIGURE 1. The 3-adic absolute value (see [H, Figure 2]).

1.3. **Facts and definitions.** Here is the promised summary.

1. For $n \in \mathbf{N}$ set

$$T_n := \left\{ \sum_{\boldsymbol{\nu} \in \mathbf{N}^n} c_{\boldsymbol{\nu}} \zeta_1^{\nu_1} \dots \zeta_n^{\nu_n} \in K[|\zeta_1, \dots, \zeta_n|] \,\middle|\, \lim_{|\boldsymbol{\nu}| \to \infty} c_{\boldsymbol{\nu}} = 0 \right\}$$

where $\zeta_1, \dots, \zeta_n$ are free analytic variables over K. Here $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ and $|\boldsymbol{\nu}| = \nu_1 + \dots + \nu_n$. T_n is called the *Tate algebra* of dimension n over K. On T_n we have a norm, the so-called *Gauss norm*, namely $\| \; \| \colon T_n \to \mathbf{R}_+$ given by

$$\Big\| \sum_{\boldsymbol{\nu} \in \mathbf{N}^n} c_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}} \Big\| = \max_{\boldsymbol{\nu}} |c_{\boldsymbol{\nu}}|.$$

2. The map $\| \ \|$ induces an ultrametric on T_n making it a *multiplicative Banach K-algebra*[3] and $\| \ \|$. This means $\| \ \|$ satisfies (a) $\|f\| = 0$ if and only if $f = 0$, (b) $\|a\| = |a|$ for all $a \in K$, (c) $\|fg\| = \|f\|\|g\|$ for $f, g \in T_n$, and (d) $\|f + g\| \leq \max(\|f\|, \|g\|)$ and T_n is complete in the resulting metric. Property (c) is the reason $\| \ \|$ is called multiplicative and (d) the reason for the term ultrametric for the induced metric on T_n. (See [RAG, Theorem 1.2.1, Lecture 4] as well as [*ibid*, p. 3, Lecture 3] for quick proofs.)

3. T_n is noetherian, of Krull dimension n [RAG, Theorem 1.2.1 Lecture 7]. It is a regular ring [RAG, Theorem 3.2.3, Lectures 9 and 10].

4. If $\mathfrak{a}$ is an ideal[4] in T_n, then we have a *Noether normalisation* for $T_n/\mathfrak{a}$, i.e. a finite injective K-algebra homomorphism

$$T_d \hookrightarrow T_n/\mathfrak{a}$$

with d (necessarily) equal to the Krull dimension of $T_n/\mathfrak{a}$ [RAG, Theorem 1.2.4 Lecture 7].

5. If $\mathfrak{a}$ is an ideal of T_n then $T_n/\mathfrak{a}$ is a *Jacobson ring*, i.e. every prime ideal in $T_n/\mathfrak{a}$ is the intersection of the maximal ideals containing it [RAG, §§2.1, Lecture 8].

6. More importantly, every ideal in T_n is closed [*ibid*, §§2.2]. This means that if $A = T_n/\mathfrak{a}$, $\mathfrak{a} \subset T_n$ an ideal, and $\alpha\colon T_n \twoheadrightarrow A$ the canonical surjection, then the map

$$\| \ \|_\alpha \colon A \to \mathbf{R}_+$$

given by

$$\|x\|_\alpha = \inf\{\|f\| \mid f \in \alpha^{-1}(x)\}, \qquad (x \in A)$$

defines a norm $\| \ \|_\alpha$ on A.

7. An *affinoid K-algebra* is a K-algebra A which is the surjective image of a K-algebra map from some Tate algebra. From **6.** above, if $\alpha\colon T_n \twoheadrightarrow A$ then we have a residue norm $\| \ \|_\alpha$ on it. If $\beta\colon T_d \twoheadrightarrow$ is another surjective K-algebra homomorphisms then $\| \ \|_\alpha$ and $\| \ \|_\beta$ are equivalent norms on A in the sense that there exist constants $C > 0$ and $C' > 0$ such that $C\|\cdot\|_\beta \leq \|\cdot\|_\alpha \leq C'\|\cdot\|_\beta$. Moreover any K-algebra map between affinoid algebras is continuous. The complete statement is:

Theorem: [RAG, Corollary 2.2.5, Lecture 11] *Let A and B be affinoid algebras endowed with residue norms arising from surjective maps from Tate algebras, and let $\varphi\colon A \to B$ be a K-algebra homomorphism. Then φ is continuous. In particular if we have two surjective homomorphisms from Tate algebras to A, say $\alpha\colon T_n \twoheadrightarrow A$ and $\beta\colon T_m \twoheadrightarrow A$, then $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are equivalent.* (See also [*ibid*, Theorem 2.2.4].)

[3] There is some confusion in the literature regarding the use of the term Banach K-alegbra. The way Berkovich uses the term *Banach ring* in [B], the requirement that $\| \ \|$ be multiplicative is not necessary. On the other hand, for a number of authors, a Banach algebra is by definition multiplicative. So, to hedge my bets, I have called T_n a multiplicative Banach algebra

[4] An ideal will mean a proper ideal, i.e., $\mathfrak{a} \neq T_n$.

8. Consider the “unit polydisc” in $\overline{K}^n$,

$$\mathbb{B}^n(\overline{K}) = \left\{ (x_1, \dots, x_n) \in \overline{K}^n \,\middle|\, |x_i| \le 1,\ 1 \le i \le n \right\}.$$

For $f \in T_n$, and $\boldsymbol{x} \in \mathbb{B}^n(\overline{K})$, $f(\boldsymbol{x})$ makes sense as an element of $\overline{K}$. Indeed, if $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{B}^n(\overline{K})$ then $K(x_1, \dots, x_n)$ is a finite extension of K, and since K is complete, so is $K(x_1, \dots, x_n)$, whence the power series f evaluated at $\boldsymbol{x}$ converges.[5] It turns out that f attains a maximum on $\mathbb{B}^n(\overline{K})$ and the following formula, the so-called *Maximum Modulus Principle*, holds (see [RAG, § 3.2, Lecture 4]:

$$\|f\| = \max \left\{ |f(\boldsymbol{x})| \,\middle|\, \boldsymbol{x} \in \mathbb{B}^n(\overline{K}) \right\}.$$

9. For $\boldsymbol{x} \in \mathbb{B}^n(\overline{K})$, let $\mathfrak{m}_{\boldsymbol{x}}$ be the kernel of the evaluation map $f \mapsto f(\boldsymbol{x})$. Then $\mathfrak{m}_{\boldsymbol{x}} \in \mathrm{Max}(T_n)$, the maximal spectrum of T_n. We thus have a map $\mathbb{B}^n(\overline{K}) \to \mathrm{Max}(T_n)$ given by $\boldsymbol{x} \mapsto \mathfrak{m}_{\boldsymbol{x}}$. It turns out the map is surjective (see [RAG, Theorem 1.3.1, Lecture 7]) and the proof of *loc.cit.* shows that the fibres of this map are the orbits of $\mathbb{B}^n(\overline{K})$ under $\mathrm{Gal}(\overline{K}/K)$. It follows that the Gauss norm on T_n can also be computed by the fomula:

$$\|f\| = \sup_{\boldsymbol{x} \in \mathrm{Max}(T_n)} |f(\boldsymbol{x})| = \max_{\boldsymbol{x} \in \mathrm{Max}(T_n)} |f(\boldsymbol{x})| \qquad (f \in T_n)$$

where $f(\boldsymbol{x})$ is the image of f in the field $K(\boldsymbol{x}) := T_n/\mathfrak{m}_{\boldsymbol{x}}$.

10. With **9.** in mind we define the *sup norm* $\| \, \|_{\mathrm{sup}}$ on an affinoid algebra A by the formula

$$\|f\|_{\mathrm{sup}} := \sup_{\boldsymbol{x} \in \mathrm{Max}(A)} |f(\boldsymbol{x})| \qquad (f \in A).$$

Here (and everywhere) we write $\mathfrak{m}_{\boldsymbol{x}}$ for $\boldsymbol{x} \in \mathrm{Max}(A)$ when we think of it as a maximal ideal, and $f(\boldsymbol{x})$ is the image of f in $K(\boldsymbol{x}) := A/\mathfrak{m}_{\boldsymbol{x}}$. It turns out that for $f \in A$:

$$\|f\|_{\mathrm{sup}} = \sup_{\boldsymbol{x} \in \mathrm{Max}(A)} |f(\boldsymbol{x})| = \max_{\boldsymbol{x} \in \mathrm{Max}(A)} |f(\boldsymbol{x})|. \tag{1.3.1}$$

Recall from noether normalisation that $K(\boldsymbol{x})$ is finite over K and hence $|\cdot|$ extends uniquely from K to $K(\boldsymbol{x})$. Note that $\| \, \|_{\mathrm{sup}}$ is an “intrinsic” function, unlike the various residue norms on A.

The sup norm need not be a norm. Indeed, it is not hard to see that $\|f^n\|_{\mathrm{sup}} = \|f\|_{\mathrm{sup}}^n$, and hence if $f \neq 0$ is nilpotent, we have $\|f\|_{\mathrm{sup}} = 0$. If A is reduced then the sup norm is indeed a norm. Here are some well-known properties of the sup norm.

(a) The sup norm $\| \, \|_{\mathrm{sup}}$ on A is a semi-norm.

(b) $\| \, \|_{\mathrm{sup}}$ is *power multiplicative*, i.e. $\|f^n\|_{\mathrm{sup}} = \|f\|_{\mathrm{sup}}^n$ for $f \in A$.

(c) If $\alpha \colon T_n \twoheadrightarrow A$ is a surjective K-algebra homomorphism, then

$$\|f\|_{\mathrm{sup}} \le \|f\|_{\alpha} \qquad (f \in A).$$

In particular the map $(A, \| \, \|_{\alpha}) \to (A, \| \, \|_{\mathrm{sup}})$ which is the identity on the underlying sets, is continuous.

(d) $\|f\|_{\mathrm{sup}} = 0$ if and only if f is nilpotent.

[5] It should be pointed out that $\overline{K}$ is not in general complete, even if K is.

(e) $\| \|_{\sup}$ is a norm if and only if A is reduced in which case it is equivalent to every residue norm on A.

The proofs are scattered over a number of lectures and homework assignments in [RAG].

2. Berkovich spectra

2.1. **Seminorms.** Let G be an abelian group. A *seminorm* on G is a function

$$\| \|\colon G \longrightarrow \mathbf{R}_+$$

such that $\|0\| = 0$ and $\|x - y\| \leq \|x\| + \|y\|$ for $x, y \in G$. It is *non-archimedean* if $\|x - y\| \leq \max\{\|x\|, \|y\|\}$. If $\|\cdot\|$ is a seminorm on G then the collection of sets $\{U_\epsilon\}_{\epsilon>0}$ defined for each $\epsilon > 0$ by

$$U_\epsilon = \{x \in G|\, \|x\| < \epsilon\}$$

forms a fundamental system of neighbourhoods at 0 giving rise to a unique topology on G.

A seminorm $\| \|$ on G is called a *norm* if $\|x\| = 0$ implies $x = 0$.

Two seminorms $\| \|$ and $\| \|'$ are said to be *equivalent* if there exist positive real numbers C and C' such that $\|x\| \leq C\|x\|'$ and $\|x\|' \leq C'\|x\|$ for every $x \in G$.

There is an obvious definition of a Cauchy sequence on a seminormed group $(G, \| \|)$. The *separated completion* $\widehat{G}$ *of* G is defined as the space of usual equivalence classes of Cauchy sequences. Then the usual theory of completing pseudometric spaces gives us a seminorm on $\widehat{G}$ and a continuous map $\kappa = \kappa_G\colon G \to \widehat{G}$ with $\kappa(G)$ dense in G. It is easy to see that the following are equivalent

(a) The topology on G is Hausdorff.
(b) $\| \|$ is a norm.
(c) The canonical map $\kappa\colon G \to \widehat{G}$ is injective.
(d) The map $G \to \kappa(G)$ induced by κ is a homeomorphism.

Let H be a subgroup of G. Define a seminorm on G/H in the usual way, viz., if $\pi\colon G \to G/H$ is the canonical surjective homomorphism, then

$$\|y\| := \inf_{x\in\pi^{-1}(y)} \|x\| \qquad (y \in G/H).$$

One checks that

- $\| \|\colon G/H \to \mathbf{R}_+$ is indeed a seminorm. It is called the *residue seminorm* on G/H.
- The residue seminorm on G/H is a norm if and only if H is closed in G.

2.2. **Banach rings.** Let A be a ring with 1 (not necessarily commutative!). A *seminorm* on A is a seminorm on $\| \|$ on $(A, +)$ such that $\|1\| = 1$, and $\|ab\| \leq \|a\|\|b\|$ for $a, b \in A$.

A seminorm on a ring A is called *power multiplicative* if $\|a^n\| = \|a\|^n$ for $a \in A$ and $n \geq 1$. A seminorm on A is called *multiplicative* if $\|ab\| = \|a\|\|b\|$ for $a, b \in A$.

$(A, \| \|)$ is called a *seminormed ring* if $\| \|$ is a seminorm on the ring A. It is a *normed ring* if $\| \|$ is a norm.

A *Banach ring* is a normed ring A that is complete with respect to its norm.

Let $(A, \| \|)$ be a Banach ring and $\mathfrak{a}$ a closed two sided of A. Then $A/\mathfrak{a}$ is a Banach ring with respect to the residue norm. It follows that a if $\mathfrak{a}$ is a maximal two sided ideal of A then it is closed since seminorms on division rings are necessarily norms.

2.3. **The spectrum of a Banach ring.** Let $(\mathscr{A}, \| \ \|)$ be a *commutative* Banach ring with identity. A seminorm $| \ |$ on $\mathscr{A}$ is said to be *bounded* if there exists $C > 0$ such that $|f| \leq C\|f\|$ for all $f \in \mathscr{A}$. The *spectrum* $\mathscr{M}(\mathscr{A})$ is the set of all bounded multiplicative seminorms on $\mathscr{A}$.[6] For each $f \in \mathscr{A}$ we have a map

$$\Psi_f \colon \mathscr{M}(\mathscr{A}) \longrightarrow \mathbf{R}_+$$

given by $| \ | \mapsto |f|$. Endow $\mathscr{M}(\mathscr{A})$ with the weakest topology such that each Ψ_f is continuous as f varies over $\mathscr{A}$. We would like to think of $\mathscr{M}(\mathscr{A})$ as a set of points and so we play the usual notational trick that we are familiar with from algebraic geometry, as well a rigid analytic geometry. Namely, if $x \in \mathscr{M}(\mathscr{A})$ and we wish to regard x as a seminorm on $\mathscr{A}$, then we write $| \ |_x$ for x.

Theorem 2.3.1. ([B, Thm. 1.2.1]) *$\mathscr{M}(\mathscr{A})$ is non-empty, Hausdorff and compact.*

We will (hopefully) prove this next time.

References

[B] V. G. Bervkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Mathematical Surveys and Monographs, no. **33**, AMS, Providence, Rhode Island, 1990.

[H] J.E. Holly, *Pictures of Ultrametric Spaces, the p-adic Numbers, and Valued Fields*, American Mathematical Monthly, October 2001. Available at https://www.colby.edu/math/faculty/Faculty_files/hollydir/Holly01.pdf.

[RAG] Course on Rigid Analytic Geometry at CMI. https://www.cmi.ac.in/~pramath/teaching.html#RAG.

[6] Note that we are considering seminorms and not just norms. On the other hand, these seminorms are multiplicative.