Arithmetic on abelian varieties

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What is an Abelian variety?

This is a report on joint work with V.K. Murty.

- An Abelian variety is an irreducible, smooth projective group variety.
- Elliptic curves (dimension 1),
- Jacobian of a curve of genus g (dimension g).
- In dimension 4 and higher, not every Abelian variety is a Jacobian.
- However, every simple Abelian variety is a quotient of a Jacobian.

The dual abelian variety

Fix an abelian variety A over a field k. The *dual abelian variety* $\widehat{A}$ is the "space" of isomorphism classes of line bundles L on A which are *algebraically equivalent* to the trivial line bundle $\mathscr{O}_A$ on A in the following sense:

- Two line bundles L_0 and L_1 on A are *algebraically equivalent* if there is a connected variety (or connected algebraic scheme) S over k, two k-rational points s_0 and s_1 of S such, and a line bundle $\mathscr{L}$ on $A \times S$ such that $\mathscr{L}|_{A\times\{s_0\}} \cong L_0$ and $\mathscr{L}|_{A\times\{s_1\}} \cong L_1$.

(Mumford)

A line bundle L on A is algebraically equivalent to $\mathscr{O}_A$ if and only if $t_a^*L \cong L$ for every $a \in A$.

Our goal is to realise $\widehat{A}$ as a subvariety of a suitable Grassmannian given *minimal* data about A. E.g. $V = \mathrm{H}^0(A, M)$ where M is a very ample line bundle, $V_1 = \mathrm{H}^0(A, M \otimes M)$, the natural map $\mu\colon V \otimes_k V \to V_1$ and some important subspaces of V.

Note: We are not given A or M or $M \otimes M$.

- The Grassmannian in which $\widehat{A}$ will be embedded will be a Grassmannian of subspaces of V of a certain fixed dimension.
- Will say more on this later.

Hilbert polynomials

Suppose X is a complete k-scheme, L_0 a line bundle on it, and $\mathscr{F}$ a coherent $\mathscr{O}_X$- module. Then the Grothendieck-Riemann-Roch theorem shows that the expression $\Psi_{\mathscr{F}}(n)(=\Psi_{\mathscr{F}}^{L_0}(n))$ for integers n, given by

$$n \mapsto \chi(X, \mathscr{F} \otimes L_0^n) \qquad (n \in \mathbb{Z})$$

is a polynomial expression in n.

Definition

The polynomial $\Psi_{\mathscr{F}} \in \mathbb{Q}[t]$ is called the *Hilbert polynomial* of $\mathscr{F}$ with respect to L_0.
If Z is a closed subscheme of X, the *Hilbert polynomial of Z* is $\Psi_{\mathscr{O}_Z}$.
We will denote it $\Phi_Z(t)$.

- Have $\deg \Psi_{\mathscr{F}} = \dim \mathrm{Supp}(\mathscr{F})$. In particular, if L is a line bundle,

$$\deg \psi_L = \dim X.$$

- If two line bundles (or for that matter closed subschemes) are algebraically equivalent then it is well known that they have the same *Hilbert polynomial* with respect to L_0. The converse is not true – the Quot scheme/Hilbert scheme for a fixed polynomial Ψ need not be connected!
- For us L_0 will be an ample line bundle, in which case $\Psi_{\mathscr{F}}(n)$ can be shown to be a polynomial in n without appealing to GRR (Hilbert showed it — long before Hirzebruch or Grothendieck).

Riemann-Roch

Fix an abelian variety A of dimension g over a field k.

- (Riemann-Roch for Abelian Varieties) Let L be a line bundle on A and D any divisor such that $\mathscr{O}(D) \cong L$. Then

$$\chi(A,\ L) = \frac{\int_A D^g}{g!}.$$

- So, with D_0 such that $L_0 \cong \mathscr{O}(D_0)$ we have:

$$\Psi_L(n) = \frac{\int_A (D + nD_0)^g}{g!}.$$

- In particular (with $\chi_0 := \chi(A,\ L_0)$):

$$\Psi_{L_0}(t) = (1+t)^g \chi_0.$$

Algebraically trivial bundles

As we pointed out, if $L \sim_{\text{alg}} L'$, then $\Psi_L = \Psi_{L'}$, but that the converse is not in general true. There are important situations (which we exploit) where this is true. First, for a line bundle L let $K(L) = \{a \in L \mid t_a^* L \cong L\}$.

Proposition

Let L be a line bundle on A, and $d = \dim K(L)$. Then
(a) (Moonen-Van der Geer) $\Psi_L(t) = t^d f(t)$, $f(0) \neq 0$, $f \in \mathbb{Q}[t]$.
(b) $L \sim_{\text{alg}} \mathscr{O}_A \iff \Psi_L(t) = \chi_0 t^g \iff \Psi_L(t) = ct^g$, with c a constant.

In particular, algebraic equivalence with the trivial bundle is completely characterised by the Hilbert polynomial.

Ample bundles

Hilbert polynomials also let us figure our when line bundles are ample on and abelian variety.

Theorem (Mumford-Kempf-Ramanujam)*

(a) L is ample if and only if all its complex roots are negative real numbers.
(b) L ample $\implies \Psi_L(n) = \mathrm{H}^0(A, L \otimes L_0^n),\ n \geq 0.$

Corollary to the last two theorems

$$L \sim_{\text{alg}} L_0 \iff \Psi_L(t) = \chi_0(1+t)^g.$$

- L a line bundle on $A \implies \Psi_{L^{-1}}(t) = (-1)^g \Psi_L(t)$
  (Use the identity $(-D+nD_0)^g = (-1)^g(D-nD_0)^g$ and GRR.)
- Suppose $L_0 = \mathcal{O}_A(H)$, where H is effective. The exact sequence

$$0 \longrightarrow \mathcal{O}(-H) \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_H \longrightarrow 0$$

  gives us the formula for $\Phi_H(t)$ (the Hilbert polynomial of H)

$$\Phi_H(t) = \chi_0[t^g - (t-1)^g].$$

- In particular, all effective divisors algebraically equivalent to H have Hilbert polynomial

$$\Phi(t) := \chi_0[t^g - (t-1)^g].$$

The embedding

- Fix $N \geq 2$ and set $V = \mathrm{H}^0(A, \mathscr{O}(NH))$.
- For $D \geq 0$ and $D \sim_{\text{alg}} H$, let

$$W_D := \mathrm{H}^0(\mathscr{O}(NH - D)) \subset V.$$

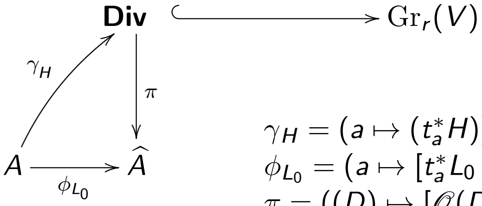
- Have $NH - D \sim_{\text{alg}} (N-1)H$. Hence
  - $NH - D$ is ample.
  - $\dim W_D = \chi_0(N-1)^g = r$ (say).
- For $N \geq 3g + 2$, the map $D \mapsto W_D$ gives an embedding

$$\mathbf{Div} \hookrightarrow \mathrm{Gr}_r(V),$$

where

- $\mathbf{Div}(= \mathbf{Div}^{L_0}) = \{D \geq 0 \mid D \sim_{\text{alg}} H\}$.
- $\mathrm{Gr}_r(V)$ = Grassmannian of r-dimensional subspaces of V

The Grothendieck-Matsusaka method



$$\gamma_H = (a \mapsto (t_a^* H));$$
$$\phi_{L_0} = (a \mapsto [t_a^* L_0 \otimes L_0^{-1}])$$
$$\pi = ((D) \mapsto [\mathscr{O}(D) \otimes L_0^{-1}])$$

(a)(Mumford) The map ϕ_{L_0} is an *isogeny*.
(b) The map π is surjective, and the fibre of π over $[L]$ is the complete linear system associated with the ample bundle $L \otimes L_0$.
(c) If L_0 is a *principal polarisation*, i.e., $\dim \mathrm{H}^0(A, L_0) = 1$, then all arrows in the above diagram are isomorphisms.

Let $N \geq 3$ and $V = \mathrm{H}^0(A,\, L_0^N) = \mathrm{H}^0(A,\, \mathscr{O}(NH))$. For each effective divisor D we have associated a subspace $W_D = \mathrm{H}^0(\mathscr{O}(NH - D)) \subset V$. One can go the other way.

Definition

Suppose $0 \neq W \subset V$ is a k-vector space. Let $\mathfrak{d}$ be the complete linear system associated with L_0^N and $\mathfrak{d}_W \subset \mathfrak{d}$ the sub-linear system given by W. Define Δ_W to be the fixed component of $\mathfrak{d}_W$.

Note that for a prime divisor Z in A,

$$\operatorname{ord}_Z \Delta_Q = \min_{s \in W \smallsetminus \{0\}} \{\operatorname{ord}_Z(s) + \operatorname{ord}_Z (NH)\}\,.$$

Fact

$0 \neq W_1 \subset W_2 \subset V \implies \Delta_{W_2} \leq \Delta_{W_1}.$

Definition

For $D \geq 0$, let

$$D^* := \Delta_{W_D}$$

Would like D to equal D^*. The exact relationship is

Proposition

Let $D \geq 0$ and F the fixed component of the complete linear system defined by $\mathscr{O}(NH - D) = L_0^N \otimes \mathscr{O}(-D)$. Then

$$D^* = D + F.$$

In particular $D^* \geq D$, and $D^* = D$ if and only if the complete linear system $|NH - D|$ has no fixed components.

In this slide, D, D' etc are effective divisors.

- $W_{D^*} = W_D$.
- $D \leq D' \implies W_D \supset W_{D'}$.
- Suppose $|NH - D'|$ has no fixed components. Then $W_D \supset W_{D'} \implies D \leq D'$.

The flip operation

When is $W = W_D$ for some $D \geq 0$? The answer involves the *flip operation*.

Definition

Suppose W is a non-zero subspace of V and $0 \neq f \in W$. Define a subspace W' $(= W'(f))$ by the formula

$$W' = \{s \in V \mid \mu(s \otimes t) \in \mu(f \otimes V), t \in W\}.$$

- Note: $f \in W'$, and so it makes sense to compute $W'' = (W'(f))'(f)$.

Fact

$W = U'$, $U \neq 0 \implies W = W_D$ for some $D \geq 0$.

More precisely, we have (all "flips" with respect to f as in the statement):

Let W be a subspace of V, $0 \neq f \in W$, of V and E the effective divisor $NH - \Delta_W + (f)$.

- Turns out, $W' = W_E$.
- In particular, if $W = W''$, then $W = W_D$ for some $D \geq 0$.
- The condition $W = W''$ can be tested using linear algebra.

Absolutely simple abelian varieties

Suppose A is absolutely simple, L a line bundle on A, D a divisor on A.

- Either $L \sim_{\text{alg}} \mathscr{O}_A$ or L is *non-degenerate*, i.e., $K(L)$ is finite.
- L ample $\iff L \not\cong \mathscr{O}_A$ and $\mathrm{H}^0(A, L) \neq 0$.
- D ample $\iff D \equiv D'$, $D' > 0$.
- $h^0(L) > 1 \implies$ the complete linear system given by L has no fixed components.

Assumptions and strategy

Recall we are trying to look for the locus of **Div** in a Grassmanian of subspaces of $V = \mathrm{H}^0(L_0^N)$. From now on we will assume

- A is *absolutely simple of dimension g* and as before
- L_0 is a *principal polarization*, i.e., $\mathrm{H}^0(L_0) = 1$.

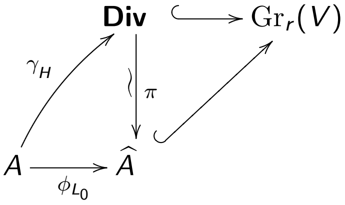
Preliminary Strategy

Let a non-zero subspace W of V be given. Pick $0 \neq f \in W$.

- Check if $W = W''$. So we know $W = W_D$ for some $D \geq 0$.
- For $j = 0, \ldots, g$ find a subspace $T_j(W)$ by linear algebraic means so that $T_j(W) = W_{D+jH}$.
- Check that $\dim T_j(W) = \chi_0(N - j - 1)^g = (-1)^g \Psi(j - N)$. This will force $\Psi_{\mathcal{O}(D)}(t)$ to equal $\Psi(t) = \chi_0(1 + t)^g$.

$$W_{D_g} \hookrightarrow W_{D_{(g-1)}} \hookrightarrow \dots \hookrightarrow W_{D_2} \hookrightarrow W_{D_1} \hookrightarrow W_D$$

$$D_j = D + jH$$



$$r = (N-1)^g$$
$$\gamma_H : a \mapsto (t_a^* D)$$
$$\phi_{L_0} : a \mapsto [t_a^* L_0 \otimes L_0^{-1}]$$

Forward and backward shifts

Goal: Given $0 \neq W \subset V$, define $T_j(W) \subset W$. $(T_j(W_D)\text{``}=\text{''}W_{D+jH})$. Fix $j \geq 0$. Assume we know the canonical map $\mu\colon V \otimes_k V \to \mathrm{H}^0(A, L_0^{2N})$, and the spaces $\mathrm{H}^0(L_0^j)$ and $\mathrm{H}^0(L_0^{N-j})$. (We only need $j = 3, 4, 5$).

- $T_j(W) := \{s \in W \mid \mu(s \otimes t) \in \mu(W \otimes_k \mathrm{H}^0(L_0^{N-j})), \forall t \in V\}$.
- $\mu_j(W) := \mu(W \otimes_k \mathrm{H}^0(L_0^j)) \subset \mathrm{H}^0(L_0^{2N})$. $(\mu_j(W_D)\text{``}=\text{''}W_{D-jH})$
- $T_j(W_D) \subset W_{D+jH}$
- $\mu_j(W_D) \subset W_{D-jH}$ provided $D - jH$ is effective.

m-regularity

- A coherent sheaf on a projective scheme $(X, \mathscr{O}(1))$ is *m-regular* if $\mathrm{H}^i(X, \mathscr{F}(m-i)) = 0$ for $i \geq 1$.
- (Mumford) If $\mathscr{F}$ is m-regular then it is $(m+1)$-regular and the natural map $\mathrm{H}^0(\mathscr{F}(m)) \otimes_k \mathrm{H}^0(\mathscr{O}(1)) \to \mathrm{H}^0(\mathscr{F}(m+1))$ is surjective.
- Returning to our abelian variety A, if L is ample then L is g-regular ($g = \dim A$). Indeed, we only have to check

$$\mathrm{H}^i(A, L(g-i)) = 0 \qquad (\text{for } 0 < i \leq g).$$

  In that range $g - i \geq 0$, whence $L \otimes \mathscr{O}(g-i)$ is ample.
- We did not need absolute simplicity of A for the above, but if it is (our running assumption), then $h^0(L) > 1$ ensures ampleness of L.

- Let $D, D' \geq 0$ with $\dim W_D > 1$. Fix $j \in \{1, \ldots, N-3\}$. Then

$$T_j(W_{D'}) = W_D \text{ and } \mu_j(W_D) = W_{D'} \implies D = D' + jH.$$

- Let $3 \leq j < \frac{N}{g+1}$. Suppose D is an effective divisor such that $L_0^{N-j} \otimes \mathscr{O}(-D)$ is 0-regular with respect to L_0^j. Then

$$T_j(W_D) = W_{D+jH} \quad \text{and} \quad \mu_j(W_{D+jH}) = W_D.$$

- The above is (of course) for A absolutely simple.
- Recall also that if $W = W''$, then $W = W_D$ for some $D \geq 0$.
- $L_0^{N-j} \otimes \mathscr{O}(-D)$ is 0-regular $\iff L_0^{N-j-g} \otimes \mathscr{O}(-D)$ is g-regular. A sufficient condition is that $h^0(L_0^{N-j-g} \otimes \mathscr{O}(-D)) > 1$. This is ensured when $T^{j+g}(W_D) > 1$.

From now on

- $N = 6g + 6,\ r = (6g + 5)^g$.
- $V = H^0(A,\ L_0^N)$.

Assumptions

Assume we know

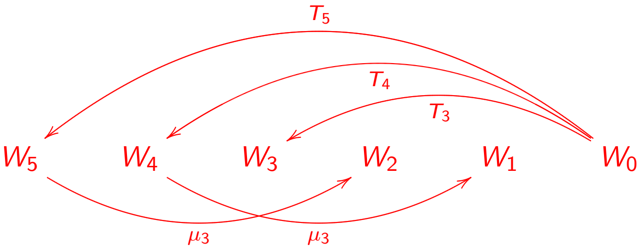
- $\mu\colon V \otimes_k V = \mathrm{H}^0(L_0^N) \otimes_k \mathrm{H}^0(L_0^N) \to \mathrm{H}^0(L_0^{2N})$,
- $\mathrm{H}^0(L_0^j)$ and $\mathrm{H}^0(L_0^{N-j})$ for $j = 3, 4, 5$.

Consider the embedding:

$$\mathrm{Gr}_r(V) \hookrightarrow \mathbf{Div}$$

Given a subspace $W \subset V$ of dimension r, i.e., a point $x_W \in \mathrm{Gr}_r(V)$, how does one decide whether $x_W \in \mathbf{Div}$ or not?

Let $W_0 = W$. Get W_1, W_2, W_3, W_4 and W_5 as follows:

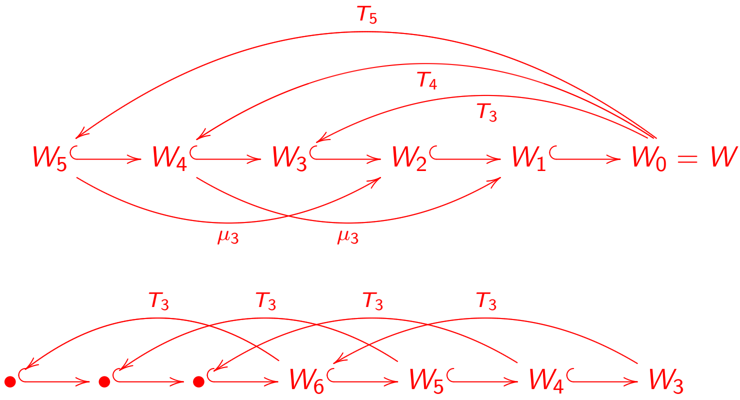


Define W_g, W_{g-1}, ..., W_6, via the formula

$$W_{i+3} = T_3(W_i) \qquad (i = 3, \ldots, g-3).$$

Impose the following conditions:

1. $W_g \subset W_{g-1} \subset \ldots \subset W_2 \subset W_1 \subset W_0$.
2. $T_3(W_1) = W_4, \; T_3(W_2) = W_5$.
3. $\mu_4(W_4) = \mu_5(W_5) = W_0$.
4. $\mu_3(W_i) = W_{i-3}, \; i = 3, \ldots, g$.
5. $\dim_k W_i = (N - 1 - i)^g = (6g + 5 - i)^g, \; i = 0, \ldots, g$.
6. $\dim_k T_3^i(W_3) = (6g + 2 - 3i)^g, \; i = 1, \ldots, g$.
7. $W_i = (W_i'(f))'(f), \; i = 1, \ldots g$, for some (and hence every) non-zero $f \in W_g$.


T5
T4
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μ3
T3
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T3
T3
W6
W5
W4
W3

Let W be an r-dimensional subspace of V and x_W the corresponding point in $\mathrm{Gr}_r(V)$. Then $x_W \in$ **Div** if and only if W satisfies conditions (1)—(7) above.

Thank you!

Addition

1. Let $F = \mu(T_3^{g+1}(W^1) \otimes_k T_3^{g+1}(W^2))$. One can show:
   - $F \subset V$. (If $W^1 = W_D$, and $W^2 = W_E$, then $F = W_{D+E}$.)
   - $\dim F = (N-2)^g = (6g+4)^g$.
   - $0 \neq \mu_3(T_5(F)) \subset F$.
2. Pick $0 \neq \varphi \in \mu_3(T_5(F))$. Use φ to compute F', i.e., $F' = F'(\varphi)$. Note that $\varphi \in F'$, whence $F' \neq 0$.
3. Pick $0 \neq \psi \in F'$ such that

$$\mu(\psi \otimes \mathrm{H}^0(\mathscr{O}(3H))) \subset (\mu_3 T_5 F)'.$$

   One can show that such a ψ exists and is unique up to a non-zero scalar multiple.

4. Let $U = \{s \in V \mid \mu(\varphi \otimes s) \in \mu(\psi \otimes F)\}$. (Theory shows that $U \subset V$.)
5. Set $W = \mu_4(T_3(U))$. Then $W = W^1 * W^2$. In other words

$$x_W = x_{W_1} + x_{W_2}.$$