Residues and Duality

Pramathanath Sastry

Chennai Mathematical Institute

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Grothendieck Duality

At its heart Grothendieck Duality is about creating a pseudo-functor $(-)^!$ on a suitable category of algebraic geometric objects (e.g., noetherian schemes, algebraic spaces, stacks . . .) such that

- For proper maps f, $f^!$ is a right adjoint to $\mathbf{R}f_*$
- For "general" maps, $f^!$ is *supposed* to be the right adjoint to $\mathbf{R}f_!$ – the direct image with proper supports.
- $f^!$ it behaves well with respect to étale localizations of the source and with respect to flat base change.

Two approaches

- The concrete approach (Grothendieck-Hartshorne). This is the approach in Hartshorne's *Residues and Duality* [RD]. Dualizing complexes and residual complexes play a major part. For a smooth map f, the functor $f^!$ is *defined* to be $f^*(-) \otimes \Omega_f^d[d]$, $d=$ relative dimension of f. Similarly definitions are given for finite maps, projective space, ... The game is to make it all hang together to form a pseudo-functor.

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- The abstract approach (Deligne-Verdier). This is the approach first started by Deligne in the appendix to [RD], but taken to a different level by Lipman, Neeman, and their collaborators. "Upper shriek" is defined by what is does, not by fiat.

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- For f *smooth* of relative dimension d, decree (by fiat)

$$f^!\mathscr{F} := f^*\mathscr{F} \overset{\mathbf{L}}{\otimes} \Omega^d_{X/Y}[d].$$

Recall $\Omega^d_{X/Y}[d]$ is the complex

$$\begin{tikzcd} & & -d^{th}\text{-spot} \arrow[d] & & & & \\ \dots \arrow[r] & 0 \arrow[r] & \Omega^d_{X/Y} \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

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$$\cdots \longrightarrow 0 \longrightarrow \overset{\substack{-d^{th}\text{-spot}\\ \downarrow}}{\Omega^d_{X/Y}} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \cdots$$

- For f *finite* set $f^! \mathscr{F} := \mathbf{R}\mathscr{H}om^\bullet_Y(\mathbf{R}f_* \mathscr{O}_X, \mathscr{F})$.

$\vdots$

Locally (in a neighbourhood of each point) f factors as follows:

closed immersion

$$\begin{tikzcd} X \arrow[r, hookleftarrow] \arrow[dr, "f"'] & U \arrow[r, hook, "i"] & Z \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

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Problem: One can have an open cover $\{U_\alpha\}$ of X, objects $\mathscr{G}_\alpha \in \mathbf{D}(U_\alpha)$ with isomorphisms $\varphi_{\alpha\beta} : \mathscr{G}_\beta|_{U_{\alpha\beta}} \xrightarrow{\sim} \mathscr{G}_\alpha|_{U_{\alpha\beta}}$ satisfying $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, and no global $\mathscr{G} \in \mathbf{D}(X)$ such that $\mathscr{G}|_{U_\alpha} \cong \mathscr{G}_\alpha, \ldots .$

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When f is proper, would like a map $\mathrm{Tr}_f \colon \mathbf{R}f_* f^! \mathscr{F} \to \mathscr{F}$ such that $(f^!\mathscr{F}, \mathrm{Tr}_f)$ represents $\mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*(-), \mathscr{F})$. More on that later.

Duality

For a proper map of schemes $f\colon X \to Y$ we want a right adjoint to $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}_{\mathrm{qc}}(Y)$. The following theorem, in greater generality than stated below, is due to Neeman (JAMS, 1996).

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Theorem (Duality)

Let $f\colon X \to Y$ be a proper map of quasi-compact quasi-separated schemes. Then $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}_{\mathrm{qc}}(Y)$ has a right adjoint

$$f^!\colon \mathbf{D}_{\mathrm{qc}}(Y) \to \mathbf{D}_{\mathrm{qc}}(X).$$

In other words, we have a co-adjoint unit (the "trace map") $\mathrm{Tr}_f\colon \mathbf{R}f_*f^! \to \mathbf{1}_{\mathbf{D}_{\mathrm{qc}}(Y)}$ inducing a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(X)}(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(Y)}(\mathbf{R}f_*\mathscr{F}, \mathscr{G}).$$

We are only interested in noetherian schemes and finite type maps. The important relation is

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(X)}(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(Y)}(\mathbf{R}f_*\mathscr{F}, \mathscr{G})$$

for $f\colon X \to Y$ proper.

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Given a map $\mathscr{F} \xrightarrow{\varphi} f^!\mathscr{G}$, with $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X)$ and $\mathscr{G} \in \mathbf{D}_{\mathrm{qc}}(Y)$, one can form the composite $\psi = \mathrm{Tr}_f(\mathscr{G}) \circ \mathbf{R}f_*(\varphi)$:

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$$\begin{tikzcd}
\mathbf{R}f_*(\mathscr{F}) \arrow[r, "\mathbf{R}f_*(\varphi)"] \arrow[dr, "\psi"'] & \mathbf{R}f_* f^!\mathscr{G} \arrow[d, "\mathrm{Tr}_f(\mathscr{G})"] \\
 & \mathscr{G}
\end{tikzcd}$$

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The assertion is that

$$\varphi \mapsto \psi$$

is bijective, giving the required isomorphism

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(X)}(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(Y)}(\mathbf{R}f_*\mathscr{F}, \mathscr{G}).$$

There is a sheaf version.

Theorem

Let $f\colon X \to Y$ be a pseudo-coherent proper map of quasi-compact separated schemes. Then $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qc}}(X) \to \mathbf{D}^+_{\mathrm{qc}}(Y)$ has a right adjoint $f^!$. Furthermore

$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om_X(x, f^!y) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_Y(\mathbf{R}f_*(x), y).$$

This fails for unbounded complexes! In other words *flat base change for* $(-)^!$ fails for unbounded complexes (for this is really open base change for "upper-shriek"). Neeman returns to this issue in a recent manuscript.

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Let f be proper and let

$$\phi\colon \mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes} f^!\mathscr{O}_Y \longrightarrow f^!$$

be defined by the commutativity of

$$\begin{tikzcd}
\mathbf{R}f_*(\mathbf{L}f^*(\mathscr{F}) \overset{\mathbf{L}}{\otimes} f^!\mathscr{O}_Y) \arrow[d, "\mathbf{R}f_*(\phi(\mathscr{F}))"'] & \mathscr{F} \overset{\mathbf{L}}{\otimes} \mathbf{R}f_*f^!\mathscr{O}_Y \arrow[l, "\sim"', "\text{proj. formula}"] \arrow[d, "\mathbf{1}\otimes\mathrm{Tr}_f"] \\
\mathbf{R}f_*f^!(\mathscr{F}) \arrow[r, "\mathrm{Tr}_f"'] & \mathscr{F}
\end{tikzcd} \qquad (\mathscr{F}\in\mathbf{D}_{\mathrm{qc}}(Y))$$

$$\left(\mathbf{R}f_*f^! \xrightarrow{\mathrm{Tr}_f} \mathbf{1}_{\mathbf{D}_{\mathrm{qc}}(Y)}\right) = \text{the co-adjoint unit}$$

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Theorem (Neeman)

Let f and ϕ be as above. The following are equivalent:

(1) $\phi\colon \mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes} f^!\mathscr{O}_Y \longrightarrow f^!$ is an isomorphism.

(2) $f^!$ commutes with small co-products.

(3) $\mathbf{R}f_*$ sends perfect complexes to perfect complexes.

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Such maps are called quasi-perfect.

Here are examples of quasi-perfect maps

- *Flat maps:* Maps of the form $f\colon X \to Y$ such that $\mathscr{O}_{X,x}$ is flat over $\mathscr{O}_{Y,f(x)}$ for every $x \in X$.
- *Regular immersions:* Closed immersions of the form $f\colon X \hookrightarrow Y$ such that for each $x \in X$, the kernel of the surjection $\mathscr{O}_{Y,x} \twoheadrightarrow \mathscr{O}_{X,x}$ is generated by a regular sequence.

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Assume (again for simplicity)

$$I := \ker(A \twoheadrightarrow B) = (t_1, \ldots, t_d),$$

with $\mathbf{t}$ a regular sequence.

Let $K_\bullet(\mathbf{t})$ and $K^\bullet(\mathbf{t}) := \mathrm{Hom}_B(K_{-\bullet}(\mathbf{t}), A)$ be the homological and the cohomological Koszul complexes on $\mathbf{t}$. Since $\mathbf{t}$ is regular:

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We thus have

$$\begin{aligned} f_*f^!\mathscr{O}_Y &= f_*\,\mathbf{R}\mathscr{H}om^\bullet_X(\mathscr{O}_X, f^!\mathscr{O}_Y) \\ &\xrightarrow{\sim} \mathbf{R}\mathscr{H}om^\bullet_Y(f_*\mathscr{O}_X, \mathscr{O}_Y) \\ &\xrightarrow{\sim} f_*\mathscr{O}_X[-d] \end{aligned}$$

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The isomorphism depends on the choice of generators $\mathbf{t}$ of I.

The invariant way of saying this is (with $\mathscr{I} = I^{\sim}$)

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One sees, more generally, that for a regular immersion $f\colon X \hookrightarrow Y$ of codimension d, we have

$$f^! \xrightarrow{\sim} \mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes}_X \wedge^d_{\mathscr{O}_X}\mathscr{N}_{X/Y}[-d]$$

where $\mathscr{N}_{X/Y}$ is the *normal bundle* of X in Y.

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- Compactifications exist (Nagata).
- $f^!$ independent of compactification (Deligne - at least for the cases he considered). This is open base change.

As things stood until very recently, most of this made sense only for bounded below complexes if f is not proper (but we do assume f is separated and of finite type, or more generally separated and essentially of finite type). The issue has to with *flat base change*, which we will review (soon).

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- Wish to remove the boundedness hypotheses.

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v^*\mathbf{R}f_*f^! \arrow[d, "\wr"'] \arrow[dr, "v^*\mathrm{Tr}_f"] & \\
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When is this an isomorphism? More precisely, what are the conditions on f, g, or E, so that $\Phi(E)$ is an isomorphism?

"Classically" one needs $E \in \mathbf{D}^{+}_{\mathrm{qc}}(Y)$ (Verdier for Y and X of finite Krull dimension; Lipman in general). In a recent, as yet unpublished, manuscript Neeman proves:

Let f be as above. Let $E \in \mathbf{D}_{\mathrm{qc}}(Y)$. Then $\Phi(E)\colon u^*f^!(E) \longrightarrow g^!u^*(E)$ is an isomorphism if one of the following holds:

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Recall: The traditional $f^!$ for such maps is from $\mathbf{D}^+_{\mathrm{qc}}(Y)$ to $\mathbf{D}^+_{\mathrm{qc}}(X)$ (unless f is proper).

Suppose $f = p \circ i = q \circ j$ are two compactifications of f. Say we have a commutative diagram with the square cartesian.

finite tor-dimension $\rightarrow$

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \overline{\overline{X}} \arrow[d, "h"] \arrow[dr, "q"] & \\
X \arrow[r, "i"'] & \overline{X} \arrow[r, "p"'] & Y
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(Proof of $i^*p^! \xrightarrow{\sim} j^*q^!$)

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(Proof of $i^*p^! \xrightarrow{\sim} j^*q^!$)

We have $i^*(C) \xrightarrow{\sim} j^*h^!(C)$ for C in the *unbounded* derived category $\mathbf{D}_{\mathrm{qc}}(\overline{X})$.

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\end{tikzcd}$$

(Proof of $i^*p^! \xrightarrow{\sim} j^*q^!$)

We have $i^*(C) \xrightarrow{\sim} j^*h^!(C)$ for C in the *unbounded* derived category $\mathbf{D}_{\mathrm{qc}}(\overline{X})$. Setting $C = p^!E$ we get

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Suppose $f = p \circ i = q \circ j$ are two compactifications of f. Say we have a commutative diagram with the square cartesian.

finite tor-dimension $\rightarrow$

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Given (i, p) and (j, q) we can always reduce to the case considered.

The isomorphisms $i^*p^! \xrightarrow{\sim} j^*q^!$ of the previous slide allow us to define (with tedious checking of compatbilities) $f^!$. In fact one has (via the results of Nayak) :

Theorem (Neeman)

Let $\mathbb{S}_e$ be the category whose objects are noetherian schemes, and the morphisms are the separated maps essentially of finite type.

- Given $f\colon X \to Y$ in $\mathbb{S}_e$ there is a well defined functor $f^!\colon \mathbf{D}_{\mathrm{qc}}(Y) \to \mathbf{D}_{\mathrm{qc}}(X)$ in the unbounded derived category.
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From now on we assume f is separated and of finite type, and $f^!$ is as above.

Verdier's isomorphism

Suppose $f\colon X \to Y$ is *smooth* of relative dimension d. Then f is flat, and hence $f^! \xrightarrow{\sim} \mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes}_X f^!\mathscr{O}_Y$.

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$$\begin{tikzcd} X \arrow[r, "\delta"] & X'' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\clubsuit"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd} \qquad (\clubsuit \text{ cartesian})$$

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Since f is smooth, δ is a regular immersion of codimension d. Note that $\mathscr{I}_\delta/\mathscr{I}_\delta^2 = \Omega^1_{X/Y}$, whence $\delta^!\mathscr{O}_{X''} \xrightarrow{\sim} (\Omega^d_{X/Y})^{-1}[-d]$

We thus have

$$\begin{aligned}
\mathscr{O}_X \xrightarrow{\sim} \delta^! p_1^! \mathscr{O}_X &\xrightarrow{\sim} \delta^! p_1^! f^* \mathscr{O}_Y \\
&\xrightarrow{\sim} \delta^! p_2^* f^! \mathscr{O}_Y \\
&\xrightarrow{\sim} \mathbf{L}\delta^* p_2^* f^! \mathscr{O}_Y \otimes_X (\Omega^d_{X/Y})^{-1}[-d] \\
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One therefore has the *Verdier isomorphism*

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Serre duality is a special case of this.

Let X be a smooth proper k variety of dimension d, and set $Y = \mathrm{Spec}\, k$ and $f\colon X \to Y$ the structure map. The isomorphism $\mathrm{Hom}_{\mathbf{D}(X)}(\mathscr{F}, f^!\mathscr{O}_Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*\mathscr{F}, \mathscr{O}_Y)$ translates to

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$$\mathrm{Ext}^i_X(\mathscr{F},\, \Omega^d_{X/k}) \xrightarrow{\sim} \mathrm{Hom}_k(\mathrm{H}^{d-i}(X,\, \mathscr{F}),\, k).$$

Let Z be a scheme. If $C^\bullet$ is a complex of $\mathscr{O}_Z$-modules concentrated in the interval $[-d, 0]$, we have an obvious map of complexes $C^\bullet \to \mathrm{H}^0(C^\bullet)$. If D is an $\mathscr{O}_Z$-module, then a map of complexes $C^\bullet \to D$ is the same as a map $\mathrm{H}^0(C^\bullet) \to D$.

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$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & C^{-d} \arrow[r] \arrow[d] & \dots \arrow[r] & C^{-1} \arrow[r] \arrow[d] & C^0 \arrow[r] \arrow[d] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & D \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

This means that if the smooth map f is proper, giving

$$\mathrm{Tr}_f(\mathscr{O}_Y) \colon \mathbf{R}f_*\Omega^d_{X/Y}[d] \to \mathscr{O}_Y$$

is equivalent to giving

$$\mathrm{R}^d f_*\Omega^d_{X/Y} \to \mathscr{O}_Y.$$

Difficulty with Verdier as a starting point

Suppose $f\colon X \to Y$ is smooth and proper of relative dimension d. Let $\int_f$ – *the Verdier trace/integral* – be defined by the commutativity of

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We do not know $\int_f$, not even when $X = \mathbb{P}^d_Y$ and $f\colon X \to Y$ the structural map $\pi = \pi_Y\colon \mathbb{P}^d_Y \to Y$ and $Y = \operatorname{Spec} k$.

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Residues and Traces via Verdier

In recent (as yet unpublished) work, Suresh Nayak and I show

- When $f = \pi_Y$, $\int_f$ is the usual map $\mathrm{R}^d\pi_*\Omega^d_\pi \to \mathscr{O}_Y$.
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$$\begin{tikzcd}
\mathrm{H}^d_Z(X, \Omega^d_f) \arrow[rr, two heads] \arrow[drr, bend right, "\mathrm{res}_Z"'] & & \mathrm{H}^d(X, \Omega^d_f) \arrow[d, "\int_f"] \\
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In the picture above $Y = \operatorname{Spec} A$. The *residue along Z*, $\mathrm{res}_Z = \mathrm{res}_{Z,f}$, is the composite indicated. $Z \to Y$ is finite dominant.

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"Residues determine $\int$"

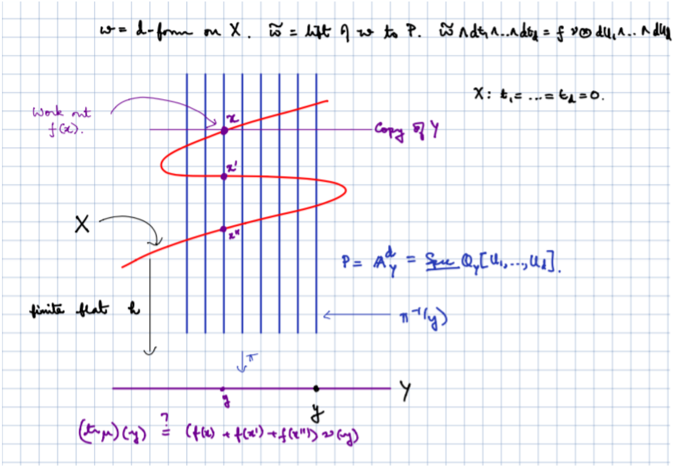
- If Z is contained in an affine open subscheme $U = \operatorname{Spec} R$ of Y, and is given up to radical by the vanishing of $t_1, \ldots, t_d$, then elements of $\mathrm{H}^d_Z(X,\,\Omega^d_f)$ can be represented by generalised fractions of the form $\begin{bmatrix} \mu \\ t_1^{\alpha_1},\ldots,t_d^{\alpha_d} \end{bmatrix}$, with $\mu \in \Omega^d_{R/A}$.
- We show that the expressions

$$\mathrm{res}_{X/Y} \begin{bmatrix} \mu \\ t_1^{\alpha_1}, \ldots, t_d^{\alpha_d} \end{bmatrix} := \mathrm{res}_Z \begin{bmatrix} \mu \\ t_1^{\alpha_1}, \ldots, t_d^{\alpha_d} \end{bmatrix}$$

satisfy all the residue formulae given in Hartshorne's [RD].

Suppose $\phi\colon X \to S$ and $\psi\colon Y \to S$ are both smooth of relative dimension d, and $h\colon X \to Y$ is a finite flat map of Y-schemes. We then have a map (from abstract nonsense and Verdier's isomorphism) $\mathrm{Tr}_h\colon h_*\Omega^d_{X/S} \to \Omega^d_{Y/S}$? Can one identify this. A possible candidate is a trace defined concretely by Lipman and Kunz and used by Kunz to build a partial theory of duality. Intuitively, here is the idea.

$\mathrm{Tr}_h \colon h_*\Omega^d_{X/S} \to \Omega^d_{Y/S}$



Verdier

For Verdier it's a different story. I remember Grothendieck had a great admiration for Verdier. He admired what we now call the Lefschetz-Verdier trace formula and Verdier's idea of defining $f^!$ first as a formal adjoint, and then calculating it later.

Bloch: I thought, maybe, that was Deligne's idea.

Illusie: No, it was Verdier's. But Deligne in the context of coherent sheaves used this idea afterward. Deligne was happy to somehow kill three hundred pages of Hartshorne's seminar in eighteen pages. (laughter)

Reminiscences of Grothendieck and his school. Luc Illusie with Alexander Beilinson, Spencer Bloch, Vladimir Drinfeld, et.al. Notices of the AMS, vol 57, no.9, Oct 2010.

" ... The abstract approach of Deligne and Verdier, and the more recent one of Neeman, seem on the surface to avoid many of the grubby details; but when you go beneath the surface to work out the concrete interpretations of the abstractly defined dualizing functors, it turns out to be not much shorter. I don't know of any royal road ... "

. J. Lipman