Residues and Duality-II

Pramathanath Sastry

Chennai Mathematical Institute

October 7, 2019
CMI Algebraic Geometry Seminar

Verdier's isomorphism

All schemes and rings in this talk are assumed noetherian for simplicity.

Recall Verdier showed, using the so called *fundamental local isomorphism* for regular immersions and the flat base change theorem for $(-)^!$, that we have an isomorphism (the Verdier isomorphism)

$$v_f\colon f^!\mathscr{O}_Y \xrightarrow{\sim} \Omega^d_{X/Y}[d]$$

when $f\colon X \to Y$ is smooth of relative dimension d. This contains within it Serre Duality for coherent sheaves on smooth proper varieties.

The Verdier trace a.k.a. integral

Suppose $f\colon X \to Y$ is smooth and proper of relative dimension d. Let $\int_f$ – *the Verdier trace/integral* – be defined by the commutativity of

$$\begin{array}{ccc} \mathrm{R}^d f_*\Omega^d_f & \xrightarrow[\mathrm{H}^0(v_f)]{\sim} & \mathrm{H}^0(\mathbf{R}f_*f^!\mathscr{O}_Y) \\ & {\scriptstyle \int_f} \searrow & \big\downarrow {\scriptstyle \mathrm{H}^0(\mathrm{Tr}_f)} \\ & & \mathscr{O}_Y \end{array}$$

The Verdier trace a.k.a. integral

Suppose $f\colon X \to Y$ is smooth and proper of relative dimension d. Let $\int_f$ – *the Verdier trace/integral* – be defined by the commutativity of

$$
\begin{array}{ccc}
\mathrm{R}^d f_*\Omega^d_f & \xrightarrow[\mathrm{H}^0(v_f)]{\sim} & \mathrm{H}^0(\mathbf{R}f_* f^{!}\mathscr{O}_Y) \\
& \searrow^{\int_f} & \downarrow{\scriptstyle \mathrm{H}^0(\mathrm{Tr}_f)} \\
& & \mathscr{O}_Y
\end{array}
$$

Goal: To describe $\int_f$ explicitly.

Goal: To describe $\int_f$ explicitly.

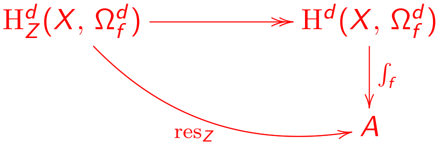
Goal: To describe $\int_f$ explicitly.
What does that mean?

Goal: To describe $\int_f$ explicitly.
What does that mean?
Need cohomology with supports, i.e. local cohomology.

Goal: To describe $\int_f$ explicitly.
What does that mean?
Need cohomology with supports, i.e. local cohomology.

$$\begin{array}{ccc} \mathrm{H}^d_Z(X,\,\Omega^d_f) & \twoheadrightarrow & \mathrm{H}^d(X,\,\Omega^d_f) \\ & \searrow^{\mathrm{res}_Z} & \downarrow{\scriptstyle\int_f} \\ & & A \end{array}$$

Goal: To describe $\int_f$ explicitly.
What does that mean?
Need cohomology with supports, i.e. local cohomology.



In the picture above $Y = \mathrm{Spec}\, A$. The *residue along Z*, $\mathrm{res}_Z = \mathrm{res}_{Z,f}$, is the composite indicated. $Z \to Y$ is finite dominant.

$$\begin{array}{ccc} \mathrm{H}^d_Z(X,\,\Omega^d_f) & \twoheadrightarrow & \mathrm{H}^d(X,\,\Omega^d_f) \\ & {\scriptstyle \mathrm{res}_Z} \searrow & \downarrow {\scriptstyle \int_f} \\ & & A \end{array}$$

"Residues determine $\int$"

$$\begin{array}{ccc} \mathrm{H}^d_Z(X,\,\Omega^d_f) & \twoheadrightarrow & \mathrm{H}^d(X,\,\Omega^d_f) \\ & \searrow^{\mathrm{res}_Z} & \Big\downarrow {\textstyle\int_f} \\ & & A \end{array}$$

"Residues determine $\int$"

If Z is contained in an affine open subscheme $U = \mathrm{Spec}\, R$ of Y, and is given up to radical by the vanishing of $t_1, \ldots, t_d$, then elements of $\mathrm{H}^d_Z(X,\,\Omega^d_f)$ can be represented by generalised fractions of the form $\left[\begin{smallmatrix} \mu \\ t_1^{\alpha_1}, \ldots, t_d^{\alpha_d} \end{smallmatrix}\right]$, with $\mu \in \Omega^d_{R/A}$.

$$\begin{array}{ccc} \mathrm{H}^d_Z(X,\,\Omega^d_f) & \twoheadrightarrow & \mathrm{H}^d(X,\,\Omega^d_f) \\ & {\scriptstyle \mathrm{res}_Z} \searrow & \downarrow {\scriptstyle \int_f} \\ & & A \end{array}$$

"Residues determine $\int$"

If Z is contained in an affine open subscheme $U = \operatorname{Spec} R$ of Y, and is given up to radical by the vanishing of $t_1, \ldots, t_d$, then elements of $\mathrm{H}^d_Z(X,\,\Omega^d_f)$ can be represented by generalised fractions of the form $\left[\begin{smallmatrix} \mu \\ t_1^{\alpha_1}, \ldots, t_d^{\alpha_d} \end{smallmatrix} \right]$, with $\mu \in \Omega^d_{R/A}$.

So really one needs to understand residues. And for that one has to first understand **generalised fractions**.

Let R be a noetherian ring and I an ideal.
Have (via excision) a long exact sequence

$$\dots \longrightarrow \mathrm{H}^i_Z(X, \mathscr{F}) \longrightarrow \mathrm{H}^i(X, \mathscr{F}) \longrightarrow \mathrm{H}^i(U, \mathscr{F}) \longrightarrow \mathrm{H}^{i+1}_Z(X, \mathscr{F}) \longrightarrow \dots$$

for $\mathscr{F}$ a sheaf of abelian groups.

Let R be a noetherian ring and I an ideal.
Have (via excision) a long exact sequence

$$\ldots \longrightarrow \mathrm{H}^i_Z(X,\mathscr{F}) \longrightarrow \mathrm{H}^i(X,\mathscr{F}) \longrightarrow \mathrm{H}^i(U,\mathscr{F}) \\ \longrightarrow \mathrm{H}^{i+1}_Z(X,\mathscr{F}) \longrightarrow \ldots$$

for $\mathscr{F}$ a sheaf of abelian groups. If $\mathscr{F}$ is quasi-coherent,

$$\mathrm{H}^i(U,\mathscr{F}) \xrightarrow{\sim} \mathrm{H}^{i+1}_Z(X,\mathscr{F}) \qquad (i \geq 1),$$

Let R be a noetherian ring and I an ideal.
Have (via excision) a long exact sequence

$$\ldots \longrightarrow \mathrm{H}^i_Z(X, \mathscr{F}) \longrightarrow \mathrm{H}^i(X, \mathscr{F}) \longrightarrow \mathrm{H}^i(U, \mathscr{F}) \longrightarrow \mathrm{H}^{i+1}_Z(X, \mathscr{F}) \longrightarrow \ldots$$

for $\mathscr{F}$ a sheaf of abelian groups. If $\mathscr{F}$ is quasi-coherent,

$$\mathrm{H}^i(U, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^{i+1}_Z(X, \mathscr{F}) \qquad (i \geq 1),$$

and

$$\mathrm{H}^0(U, \mathscr{F}) \xrightarrow{\pi} \mathrm{H}^1_Z(X, \mathscr{F}).$$

In particular (with $\mathscr{F} \in X_{qc}$) one has a surjection

$$\mathrm{H}^{d-1}(U, \mathscr{F}) \xrightarrow{\pi} \mathrel{\mkern-14mu}\rightarrow \mathrm{H}^d_Z(X, \mathscr{F})$$

for $d \geq 1$ which is an isomorphism when $d \geq 2$.

In particular (with $\mathscr{F} \in X_{qc}$) one has a surjection

$$\mathrm{H}^{d-1}(U, \mathscr{F}) \xrightarrow{\pi} \mathrm{H}^{d}_{Z}(X, \mathscr{F})$$

for $d \geq 1$ which is an isomorphism when $d \geq 2$.

Let R be a ring and I an ideal generated, up to radical, by $t_1, \ldots, t_d \in R$, and Z the closed subscheme defined by I, i.e. $Z = \mathrm{Spec}\, A/I$.

In particular (with $\mathscr{F} \in X_{qc}$) one has a surjection

$$\mathrm{H}^{d-1}(U, \mathscr{F}) \xrightarrow{\pi} \twoheadrightarrow \mathrm{H}^d_Z(X, \mathscr{F})$$

for $d \geq 1$ which is an isomorphism when $d \geq 2$.

Let R be a ring and I an ideal generated, up to radical, by $t_1, \ldots, t_d \in R$, and Z the closed subscheme defined by I, i.e. $Z = \operatorname{Spec} A/I$.

Let $X = \operatorname{Spec} R$, $U_i = \{t_i \neq 0\}$, and $\mathscr{U} = \{U_i\}$.

In particular (with $\mathscr{F} \in X_{qc}$) one has a surjection

$$\mathrm{H}^{d-1}(U, \mathscr{F}) \xrightarrow{\pi} \mathrm{H}^d_Z(X, \mathscr{F})$$

for $d \geq 1$ which is an isomorphism when $d \geq 2$.

Let R be a ring and I an ideal generated, up to radical, by $t_1, \ldots, t_d \in R$, and Z the closed subscheme defined by I, i.e. $Z = \operatorname{Spec} A/I$.

Let $X = \operatorname{Spec} R$, $U_i = \{t_i \neq 0\}$, and $\mathscr{U} = \{U_i\}$.

$\mathscr{U}$ is an affine open cover of $U = X \smallsetminus Z$.

Since $\mathscr{U}$ is an affine open cover of the separated scheme $U = X \smallsetminus Z$, and $\mathscr{F} \in X_{qc}$, we have a natural isomorphism

$$\check{\mathrm{H}}^{d-1}(\mathscr{U}, \mathscr{F}) \xrightarrow{\ \sim\ } \mathrm{H}^{d-1}(U, \mathscr{F}).$$

Since $\mathscr{U}$ is an affine open cover of the separated scheme $U = X \smallsetminus Z$, and $\mathscr{F} \in X_{qc}$, we have a natural isomorphism

$$\check{\mathrm{H}}^{d-1}(\mathscr{U}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^{d-1}(U, \mathscr{F}).$$

Now $\check{C}^{d-1}(\mathscr{U}, \mathscr{F}) = M_{t_1 \ldots t_d}$ where $M = \Gamma(X, \mathscr{F})$ is the R-module corresponding to $\mathscr{F}$.

Since $\mathscr{U}$ is an affine open cover of the separated scheme $U = X \smallsetminus Z$, and $\mathscr{F} \in X_{qc}$, we have a natural isomorphism

$$\check{\mathrm{H}}^{d-1}(\mathscr{U}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^{d-1}(U, \mathscr{F}).$$

Now $\check{C}^{d-1}(\mathscr{U}, \mathscr{F}) = M_{t_1 \ldots t_d}$ where $M = \Gamma(X, \mathscr{F})$ is the R-module corresponding to $\mathscr{F}$. We have a surjection

$$M_{t_1 \ldots t_d} = \check{C}^{d-1}(\mathscr{U}, \mathscr{F}) \xrightarrow{\pi} \check{\mathrm{H}}^{d-1}(\mathscr{U}, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^{d-1}(U, \mathscr{F}) \xrightarrow{\pi} \mathrm{H}^{d}_{Z}(X, \mathscr{F}).$$

Definition of a generalised fraction

Have $M_{t_1 \ldots t_d} \twoheadrightarrow \mathrm{H}^d_Z(X, \mathscr{F}) = \mathrm{H}^d_I(M)$.

Definition of a generalised fraction

Have $M_{t_1 \ldots t_d} \twoheadrightarrow \mathrm{H}^d_Z(X, \mathscr{F}) = \mathrm{H}^d_I(M)$. The image of $\frac{m}{t_1^{e_1} \ldots t_d^{e_d}}$ in $\mathrm{H}^d_I(M)$ under the above surjection is denoted by the generalised fraction

$$\begin{bmatrix} m \\ t_1^{e_1}, \ldots, t_d^{e_d} \end{bmatrix} \in \mathrm{H}^d_I(M).$$

Definition of a generalised fraction

Have $M_{t_1 \ldots t_d} \twoheadrightarrow \mathrm{H}^d_Z(X, \mathscr{F}) = \mathrm{H}^d_I(M)$. The image of $\frac{m}{t_1^{e_1} \ldots t_d^{e_d}}$ in $\mathrm{H}^d_I(M)$ under the above surjection is denoted by the generalised fraction

$$\begin{bmatrix} m \\ t_1^{e_1}, \ldots, t_d^{e_d} \end{bmatrix} \in \mathrm{H}^d_I(M).$$

There is a calculus of generalised fractions. For example, it turns out that if $\mathbf{s} = \{s_1, \ldots, s_d\}$ is related to $\mathbf{t}$ by the equation $s_i = \sum_j a_{ij} t_j$, then

$$\begin{bmatrix} m \\ t_1, \ldots, t_d \end{bmatrix} = \begin{bmatrix} \det(a_{ij}) m \\ s_1, \ldots, s_d \end{bmatrix}.$$

Definition of a generalised fraction

Have $M_{t_1\ldots t_d} \twoheadrightarrow \mathrm{H}^d_Z(X, \mathscr{F}) = \mathrm{H}^d_I(M)$. The image of $\frac{m}{t_1^{e_1}\ldots t_d^{e_d}}$ in $\mathrm{H}^d_I(M)$ under the above surjection is denoted by the generalised fraction

$$\begin{bmatrix} m \\ t_1^{e_1}, \ldots, t_d^{e_d} \end{bmatrix} \in \mathrm{H}^d_I(M).$$

There is a calculus of generalised fractions. For example, it turns out that if $\mathbf{s} = \{s_1, \ldots, s_d\}$ is related to $\mathbf{t}$ by the equation $s_i = \sum_j a_{ij} t_j$, then

$$\begin{bmatrix} m \\ t_1, \ldots, t_d \end{bmatrix} = \begin{bmatrix} \det(a_{ij})m \\ s_1, \ldots, s_d \end{bmatrix}.$$

This is best understood via **stable Koszul complexes**.

Stable Koszul complexes

The complex

$$K_\infty^\bullet(\mathbf{t},\, M) := \varinjlim_n K^\bullet(\mathbf{t}^n,\, M)$$

is called *the stable Koszul complex on M with respect to* $\mathbf{t}$. It is well-known that $K_\infty^\bullet(\mathbf{t},\, M)$ looks like this ↓

$$\overset{\bullet \longrightarrow \text{Čech complex} \longrightarrow}{0 \longrightarrow M \longrightarrow \check{C}^0(\mathscr{U},\mathscr{F}) \longrightarrow \check{C}^1(\mathscr{U},\mathscr{F}) \longrightarrow \dots \longrightarrow \check{C}^{d-1}(\mathscr{U},\mathscr{F}) \longrightarrow 0}$$

Formulae like $\left[\begin{smallmatrix} m \\ t_1,\dots,t_d \end{smallmatrix}\right] = \left[\begin{smallmatrix} \det(a_{ij})m \\ s_1,\dots,s_d \end{smallmatrix}\right]$ are best proven using stable Koszuls, since one can compare $K^\bullet(\mathbf{t},\, M)$ with $K^\bullet(\mathbf{s},\, M)$ and then take direct limits.

This is seen as follows. We have a map of Koszul complexes $\varphi^\bullet\colon K^\bullet(\mathbf{t},\, M) \to K^\bullet(\mathbf{s},\, M))$ lifting the identity on M, and which is multiplication by $\det(a_{ij})$ in degree d. Let $\mathscr{K}^i = K^{i+1}(\mathbf{t},\, M)^\sim$ and $\mathscr{K}'^i = K^{i+1}(\mathbf{s},\, M)^\sim$. We then have maps of complexes $\mathscr{F}[0] \to \mathscr{K}^\bullet$ and $\mathscr{F}[0] \to \mathscr{K}'^\bullet$, such that

$$\mathscr{K}^\bullet \xrightarrow{\text{via } \varphi^\bullet} \mathscr{K}'^\bullet$$

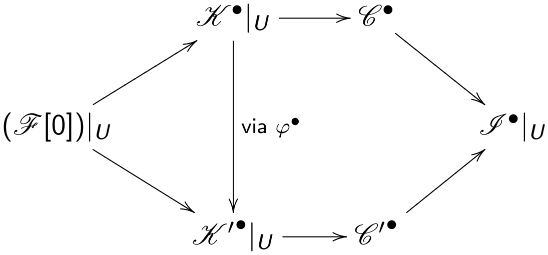
lifts the identity map on $\mathscr{F}[0]$.

This is seen as follows. We have a map of Koszul complexes $\varphi^\bullet\colon K^\bullet(\mathbf{t},\, M) \to K^\bullet(\mathbf{s},\, M))$ lifting the identity on M, and which is multiplication by $\det(a_{ij})$ in degree d. Let $\mathscr{K}^i = K^{i+1}(\mathbf{t},\, M)^\sim$ and $\mathscr{K}'^i = K^{i+1}(\mathbf{s},\, M)^\sim$. We then have maps of complexes $\mathscr{F}[0] \to \mathscr{K}^\bullet$ and $\mathscr{F}[0] \to \mathscr{K}'^\bullet$, such that

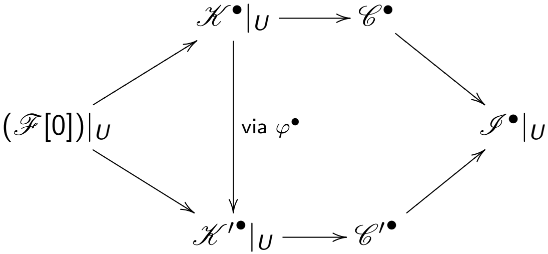
$$\mathscr{K}^\bullet \xrightarrow{\text{via } \varphi^\bullet} \mathscr{K}'^\bullet$$

lifts the identity map on $\mathscr{F}[0]$.

Let $U'_i = \{s_i \neq 0\}$ and $\mathscr{U}' = \{U'_i\}$. Let $\mathscr{C}^\bullet$ and $\mathscr{C}'^\bullet$ be the sheaf Čech complexes of $\mathscr{F}|_U$ with respect to $\mathscr{U}$ and $\mathscr{U}'$ respectively. Both are resolutions of $\mathscr{F}|_U$ and if $\mathscr{I}^\bullet$ is a quasi-coherent injective resolution of $\mathscr{F}$, the data fits into a homotopy commutative diagram as follows.



Since the map $K^d(\mathbf{t},\, M) \to K^d_\infty(\mathbf{t},\, M)$ is $m \mapsto \frac{m}{t_1 \ldots t_d}$, we conclude that the image of $\frac{m}{t_1 \ldots t_d} \in C^{d-1}(\mathscr{U},\, \mathscr{F})$ and that of $\frac{\det(a_{ij})m}{s_1 \ldots s_d} \in C^{d-1}(\mathscr{U}',\, \mathscr{F})$ in $\mathrm{H}^{d-1}(U,\, \mathscr{F})$ are the same. The assertion follows.



Since the map $K^d(\mathbf{t}, M) \to K^d_\infty(\mathbf{t}, M)$ is $m \mapsto \frac{m}{t_1 \ldots t_d}$, we conclude that the image of $\frac{m}{t_1 \ldots t_d} \in C^{d-1}(\mathscr{U}, \mathscr{F})$ and that of $\frac{\det(a_{ij})m}{s_1 \ldots s_d} \in C^{d-1}(\mathscr{U}', \mathscr{F})$ in $\mathrm{H}^{d-1}(U, \mathscr{F})$ are the same. The assertion follows. See *Dualizing sheaves, differentials forms and residues on algebraic varieties* by Joseph Lipman, Asterisque **117**, Société Mathematique de France (1984) for further details.

Residues again

Once again suppose $f\colon X \to Y = \operatorname{Spec} A$ is smooth, separated, of relative dimension d. Suppose Z is a closed subscheme of X such that $Z \to Y$ is finite and flat.

One can define $\operatorname{res}_Z\colon \mathrm{H}^d_Z(X, \Omega^d_{X/Y}) \to A$ without assuming f is proper in the following way.

Residues again

Once again suppose $f\colon X \to Y = \operatorname{Spec} A$ is smooth, separated, of relative dimension d. Suppose Z is a closed subscheme of X such that $Z \to Y$ is finite and flat.

One can define $\operatorname{res}_Z\colon \mathrm{H}^d_Z(X, \Omega^d_{X/Y}) \to A$ without assuming f is proper in the following way.

Pick a compactification $(j, \overline{X}, \bar{f})$ of f, i.e. $j\colon X \to \overline{X}$ is an open immersion and $\bar{f}\colon \overline{X} \to Y$ is proper and $\bar{f} \circ j = f$. We have a map $\operatorname{Tr}_{f,Z}\colon \mathbf{R}\Gamma_Z(X, \Omega^d_{X/Y}[d]) \to A$ defined by the commutativity of

$$
\begin{array}{ccc}
\mathbf{R}\Gamma_Z(X, \Omega^d_{X/Y}[d]) & \xrightarrow{\ \sim\ } & \mathbf{R}\Gamma_Z(X, f^!\mathscr{O}_Y) \\
 & & \big\downarrow \wr \\
\Big\downarrow \mathrm{Tr}_{f,Z} & & \mathbf{R}\Gamma_Z(\overline{X}, \bar{f}^!\mathscr{O}_Y) \\
 & & \big\downarrow \\
A & \xleftarrow[\mathrm{Tr}_{\bar{f}}]{} & \mathbf{R}\Gamma(\overline{X}, \bar{f}^!\mathscr{O}_Y)
\end{array}
$$

One shows $\mathrm{Tr}_{f,Z}$ is independent of the compactification $\bar{f}$. Define $\mathrm{res}_Z \colon \mathrm{H}^d_Z(X, \Omega^d_{X/Y}) \to A$ by

$$\mathrm{res}_Z := \mathrm{H}^0(\mathrm{Tr}_{f.Z}).$$

Assume for simplicity that Z factors through an affine open set $U = \operatorname{Spec} R$ of X and the defining ideal of Z in R is $I = (t_1, \ldots, t_d)$.

There are a number of residue formulas one needs to establish for this residue theory via Verdier's isomorphism to bring it in line with the formulas (R1)–(R10) of Grothendieck given in Hartshorne's *Residues and Duality* and proved in Conrad's *Grothendieck Duality and Base Change*, Springer LNM 1750 (2000). For example, if the finite flat map $Z \to Y$ is an isomorphism then one has to show

$$\operatorname{res}_Z \begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d \\ t_1^{e_1}, \ldots, t_d^{e_d} \end{bmatrix} = \begin{cases} 1 & \text{if } (e_1, \ldots, e_d) = (1, \ldots, 1) \\ 0 & \text{otherwise} \end{cases}$$

Thom Class

In the above situation (i.e. $Z \xrightarrow{\sim} Y$) it should be pointed out that if $s_1, \ldots, s_d \in R$ is another set of elements defining Z, then our determinant formula gives (using the fact that fractions of the form $\left[\begin{smallmatrix} m \\ t_1,\ldots,t_d \end{smallmatrix}\right]$ are annihilated by elements from the ideal $(t_1, \ldots, t_d)$):

$$\begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d \\ t_1, \ldots, t_d \end{bmatrix} = \begin{bmatrix} \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_d \\ s_1, \ldots, s_d \end{bmatrix}.$$

One should regard $\left[\begin{smallmatrix} \mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_d \\ t_1,\ldots,t_d \end{smallmatrix}\right] \in \mathrm{H}^d_Z(X, \Omega^d_{X/Y})$ as the Thom class of the normal bundle of Z in X and its image in $\mathrm{H}^d(X, \Omega^d_{X/Y})$, at least when $X \to Y$ is proper, as a (relative) fundamental class of Z in X over Y. Which is why (morally) it "integrates" to 1.