Grothendieck Duality - the abstract approach

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Grothendieck Duality

At its heart Grothendieck Duality is about creating a pseudo-functor $(-)^!$ on a suitable category of algebraic geometric objects (e.g., noetherian schemes, algebraic spaces, stacks ...) such that

- For proper maps f, $f^!$ is a right adjoint to $\mathbf{R}f_*$
- For "general" maps, $f^!$ is *supposed* to be the right adjoint to $\mathbf{R}f_!$ –the direct image with proper supports.
- $f^!$ it behaves well with respect to étale localizations of the source and with respect to flat base change.

Two approaches

- The concrete approach (Grothendieck-Hartshorne). This is the approach in Hartshorne's *Residues and Duality* [RD]. Dualizing complexes and residual complexes play a major part. For a smooth map f, the functor $f^!$ is *defined* to be $f^*(-) \otimes \Omega_f^d[d]$, $d=$ relative dimension of f. Similarly definitions are given for finite maps, projective space, ... The game is to make it all hang together to form a pseudo-functor.

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- The abstract approach (Deligne-Verdier). This is the approach first started by Deligne in the appendix to [RD], but taken to a different level by Lipman, Neeman, and their collaborators. "Upper shriek" is defined by what is does, not by fiat.

Papers that will be talked about

I will concentrate on the following papers of Neeman:

- *Grothendieck's Duality Theorem via Bousfield's techniques and Brown representability*, JAMS (1996).
- (With Lipman) *Quasi-perfect scheme-maps and boundedness of the twisted inverse image*, Illinois J. Math. (2007).
- *An improvement on the base-change theorem and the functor $f^!$*, arXiv:1406.7599.
- (With Lipman) *Fundamental Class and Verdier*, to appear in Algebraic Geometry Foundation Compositio Mathematica.

We begin with the first paper.

Compact objects

We begin with a basic definition.

Definition

Let $\mathscr{T}$ be a triangulated category closed under small co-products.

(i) An object c of $\mathscr{T}$ is *compact* if

$$\mathrm{Hom}_{\mathscr{T}}(c, \coprod_{\lambda} x_\lambda) = \coprod_{\lambda} \mathrm{Hom}_{\mathscr{T}}(c, x_\lambda)$$

for small co-products $\coprod_\lambda x_\lambda$ in $\mathscr{T}$.

(ii) $\mathscr{T}^c$ = full subcategory of $\mathscr{T}$ consisting of compact objects.

Compactly generated categories

Definition

A triangulated category $\mathscr{S}$ is *compactly generated* if small coproducts exist in $\mathscr{S}$ and there exists a subset S of compact objects in $\mathscr{S}$ satisfying one the following two equivalent conditions:

(a) $\mathrm{Hom}(s, y) = 0$ for all $s \in S \implies y = 0$.

(b) Any localizing subcategory of $\mathscr{S}$ containing S must be $\mathscr{S}$. [A localizing subcategory $\mathscr{R}$ of $\mathscr{S}$ is a full subcategory containing zero which is closed under coproducts and triangles.]

The equivalence of the two conditions (a) and (b) is not straightforward.

Here is how one produces right adjoints.

Theorem (Brown Representability)

Let $F\colon \mathscr{S} \to \mathscr{T}$ be a triangulated functor such that

(a) $\mathscr{S}$ is *compactly generated*.

(b) F respects coproducts. ($\mathscr{T}$ need not have co-products.)

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A related and very useful result is the following.

Theorem

Suppose $F\colon \mathscr{S} \to \mathscr{T}$ satisfies the requirements of Brown representability and $G\colon \mathscr{T} \to \mathscr{S}$ is a right adjoint of F. Assume $\mathscr{T}$ has small coproducrs. Then G respects coproducts if and only if for every compact object s in $\mathscr{S}$, $F(s)$ is compact in $\mathscr{T}$.

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Theorem

Suppose X and Y are quasi-compact and quasi- separated schemes. Then

(a) $\mathbf{D}_{\mathrm{qc}}(X)^c$ = perfect complexes.

(b) $\mathbf{D}_{\mathrm{qc}}(X)$ is compactly generated (in fact by a *single* perfect complex).

(c) If $f\colon X \to Y$ is a morphism of schemes, then $\mathbf{R}f_*$ commutes with arbitrary direct sums.

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Note: Neeman (1996) assumed (in addition to quasi-compactness) that X, Y and f are separated to prove this. He also proved the statement outside the parenthesis in (b). Bondal and van den Berg proved the stronger statements.

Some references

- Bondal and van den Berg: *Generators and representability of functors in commutative and non-commutative geometry*, Moscow Math. J. (2003)).
- One can also find a proof that $\mathbf{R}f_*$ respects coproducts in [Lipman, LNM 1960].

Duality

Theorem (Duality)

Let $f\colon X \to Y$ be a map of quasi-compact quasi-separated schemes. Then $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}_{\mathrm{qc}}(Y)$ has a right adjoint

$$f^\times\colon \mathbf{D}_{\mathrm{qc}}(Y) \to \mathbf{D}_{\mathrm{qc}}(X).$$

In other words, we have a co-adjoint unit (the "trace map") $\mathrm{Tr}_f\colon \mathbf{R}f_*f^\times \to \mathbf{1}_{\mathbf{D}_{\mathrm{qc}}(Y)}$ inducing a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(X)}(\mathscr{F}, f^\times\mathscr{G}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}(Y)}(\mathbf{R}f_*\mathscr{F}, \mathscr{G}).$$

There is a sheaf version.

Theorem

Let $f\colon X \to Y$ be a pseudo-coherent proper map of quasi-compact separated schemes. Then $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qc}}(X) \to \mathbf{D}^+_{\mathrm{qc}}(Y)$ has a right adjoint $f^!$. Furthermore

$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om_X(x, f^!y) \xrightarrow{\sim} \mathbf{R}\mathscr{H}om_Y(\mathbf{R}f_*(x), y).$$

This fails for unbounded complexes! In other words *flat base change for* $(-)^!$ fails for unbounded complexes. Neeman returns to this issue in a recent manuscript.

$\mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes} f^\times \mathcal{O}_Y$ versus $f^\times$

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Let

$$\phi\colon \mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes} f^\times \mathscr{O}_Y \longrightarrow f^\times$$

be defined by the commutativity of

$$\begin{array}{ccc}
\mathbf{R}f_*(\mathbf{L}f^*(\mathscr{F}) \overset{\mathbf{L}}{\otimes} f^\times \mathscr{O}_Y) & \xleftarrow[\text{proj. formula}]{\sim} & \mathscr{F} \overset{\mathbf{L}}{\otimes} \mathbf{R}f_* f^\times \mathscr{O}_Y \\
{\scriptstyle \mathbf{R}f_*(\phi(\mathscr{F}))}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{1}\otimes \mathrm{Tr}_f} \\
\mathbf{R}f_* f^\times(\mathscr{F}) & \xrightarrow[\mathrm{Tr}_f]{} & \mathscr{F}
\end{array} \qquad (\mathscr{F}\in \mathbf{D}_{\mathrm{qc}}(Y))$$

$$\left(\mathbf{R}f_* f^\times \xrightarrow{\mathrm{Tr}_f} \mathbf{1}_{\mathbf{D}_{\mathrm{qc}}(Y)}\right) = \text{the co-adjoint unit}$$

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Theorem (Neeman)

Let f and ϕ be as above. The following are equivalent:

(1) $\phi\colon \mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes} f^\times \mathscr{O}_Y \longrightarrow f^\times$ is an isomorphism.

(2) $f^\times$ commutes with small co-products.

(3) $\mathbf{R}f_*$ sends perfect complexes to perfect complexes.

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(2) $\iff$ (3) is from general principles.
(1) $\implies$ (2) Obvious.
(3) $\implies$ (1) Check $\phi(E)$ is an isomorphism for E perfect.

Quasi-perfect maps

Definition

A map $f\colon X \to Y$ between quasi-compact quasi-separated schemes is said to be *quasi-perfect* if it satisfies any of the equivalent conditions of the above Theorem.

- $\phi\colon \mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes} f^\times \mathcal{O}_Y \xrightarrow{\sim} f^\times$
- $f^\times$ commutes with small co-products
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From now on, all schemes are quasi-compact and quasi-separated (so all maps are *concentrated*).

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We need some definitions.

Pseudo-coherence

1) Complex n-pseudo-coherent if locally:

• ——— free of finite rank ———→

$$\dots \longrightarrow E^{n-1} \longrightarrow E^{n} \longrightarrow E^{n+1} \longrightarrow \dots \longrightarrow E^{n+k}$$

Pseudo-coherent $=$ n-pseudo-coherent $\forall n$.

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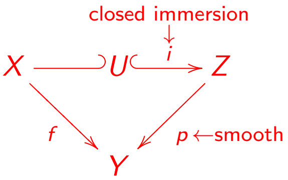
2) Map pseudo-coherent if locally (U open in X)



with $i_*\mathscr{O}_U$ pseudo-coherent

Definitions

- A map of schemes $f\colon X \to Y$ is *perfect* if it is pseudo-coherent and of finite tor-dimension.
- A map f is *quasi-proper* if $\mathbf{R}f_*$ sends pseudo-coherent complexes to pseudo-coherent complexes.

Theorem (Kiehl, 1972)

A proper pseudo-coherent map is quasi-proper.

Theorem (Lipman-Neeman, 2006)

For a map $f\colon X \to Y$ the following are equivalent:

(a) f is quasi-perfect (resp. perfect)

(b) f is quasi-proper (resp. pseudo-coherent) and of finite tor-dimension.

(c) f is quasi-proper (resp. pseudo-coherent) and $f^{\times}$ is bounded.

Observe that the definition of perfect gives us (b) $\iff$ (a) for the 'resp. case'.

Interesting features of (a) $\implies$ (c)

- Any pseudo-coherent complex can be arbitrarily well approximated *globally* by a perfect complex. This was previously known only for divisorial schemes.
- Recall that Bondal and van den Bergh proved that $\mathbf{D}_{\mathrm{qc}}(X)$ is generated by a single element. This statement is refined.

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If S is a perfect generator of $\mathbf{D}_{\mathrm{qc}}(X)$, $\exists\ A = A(S)$ such that if $E \in \mathbf{D}_{\mathrm{qc}}(X)$ with $H^j(E) \neq 0$, there exists a non-trivial map $S \to E[n]$ for some $n \geq j - A$.

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As for (c) $\implies$ (b), the crucial fact proven is: *if $f^\times$ is bounded then f is of finite tor-dimension*.

Suppose f is a proper map of noetherian schemes. The following are equivalent

- f is quasi-perfect.
- f is perfect.
- $f^{\times}$ is bounded.

Upper shriek

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We now consider only noetherian schemes.

- $f^! := f^\times$ when f is proper.
- f separated and finite type (essentially finite type enough) then *choose* a compactification $f = p \circ i$ (i.e., i an open immersion and p a proper map) and set

$$f^! := i^* p^\times$$

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- Compactifications exist (Nagata).
- $f^!$ independent of compactification (Deligne - at least for the cases he considered).

As things stood until very recently, most of this made sense only for bounded below complexes if f is not proper (but we do assume f is separated and of finite type, or more generally separated and essentially of finite type). The issue has to with *flat base change*, which we will review (soon).

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- Have:

$$f^! \colon \mathbf{D}^+_{\mathrm{qc}}(Y) \longrightarrow \mathbf{D}^+_{\mathrm{qc}}(X).$$

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- Wish to remove the boundedness hypotheses.

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The map $\mu\colon \mathbf{R}g_*u^*f^! \to v^*$ induces

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When is this an isomorphism? More precisely, what are the conditions on f, g, or E, so that $\Phi(E)$ is an isomorphism?

"Classically" one needs $E \in \mathbf{D}^{+}_{\mathrm{qc}}(Y)$ (Verdier for Y and X of finite Krull dimension; Lipman in general). In a recent, as yet unpublished, manuscript Neeman proves:

Let f be as above. Let $E \in \mathbf{D}_{\mathrm{qc}}(Y)$. Then $\Phi(E)\colon u^{*}f^{\times}(E) \longrightarrow g^{\times}u^{*}(E)$ is an isomorphism if one of the following holds:

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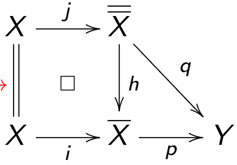
Recall: The traditional $f^!$ for such maps is from $\mathbf{D}^+_{\mathrm{qc}}(Y)$ to $\mathbf{D}^+_{\mathrm{qc}}(X)$ (unless f is proper).

Suppose $f = p \circ i = q \circ j$ are two compactifications of f. Say we have a commutative diagram with the square cartesian.

finite tor-dimension $\rightarrow$



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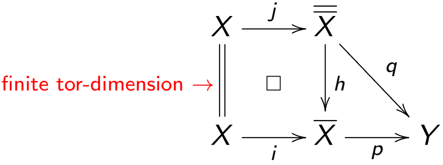

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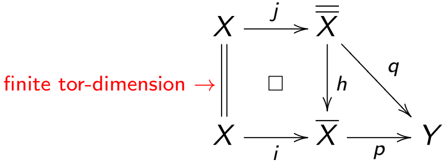

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$$i^* p^!(E) \xrightarrow{\sim} j^* h^! p^!(E) \xrightarrow{\sim} j^* q^!(E) \qquad (E \in \mathbf{D}_{\mathrm{qc}}(Y)).$$

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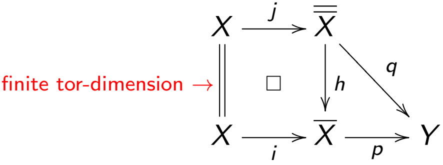

(Proof of $i^*p^! \xrightarrow{\sim} j^*q^!$)

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$$i^*p^!(E) \xrightarrow{\sim} j^*h^!p^!(E) \xrightarrow{\sim} j^*q^!(E) \qquad (E \in \mathbf{D}_{\mathrm{qc}}(Y)).$$

Given (i, p) and (j, q) we can always reduce to the case considered.

The isomorphisms $i^*p^! \xrightarrow{\sim} j^*q^!$ of the previous slide allow us to define (with tedious checking of compatbilities) $f^!$. In fact one has (via the results of Nayak) :

Theorem (Neeman)

Let $\mathbb{S}_e$ be the category whose objects are noetherian schemes, and the morphisms are the separated maps essentially of finite type.

- Given $f\colon X \to Y$ in $\mathbb{S}_e$ there is a well defined functor $f^!\colon \mathbf{D}_{\mathrm{qc}}(Y) \to \mathbf{D}_{\mathrm{qc}}(X)$ in the unbounded derived category.
- The resulting "variance theory" $(-)^!$ on $\mathbb{S}_e$ is a pseudofunctor.
- There is a map of variance theories $(-)^\times \longrightarrow (-)^!$.

The mansucript in question is:
Neeman, *An improvement on the base-change theorem and the functor* $f^!$, arXiv:1406.7599.

The Fundamental class

For an embedding of varieties $X \hookrightarrow P$ over a field k, $\dim X = d$, $\dim P = N$, P smooth, to give the fundamental class

$$[X] \in \mathrm{H}_X^{N-d}(P, \Omega_{P/k}^{N-d})$$

is to give a map (π_X=structural map)

$$c_X : \Omega^d_{X/k}[d] \to \pi_X^! k$$

which is an isomorphism on the smooth locus. This idea goes back to Grothendieck and is developed by El Zein (over **C**)(1978) and Lipman (1984).

More generally (Alonso-Jeremías-Lipman (2014)): Let $f\colon X \to Y$ in $\mathbb{S}_e$ be flat and equidimensional of relative dimension d.

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$$\begin{array}{ccccc} X & \xrightarrow{\ \delta\ } & X'' & \xrightarrow{\ p_2\ } & X \\ & & {\scriptstyle p_1}\big\downarrow & \clubsuit & \big\downarrow{\scriptstyle f} \\ & & X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

($\clubsuit$ cartesian)

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Since δ_* is left-adjoint to $\delta^!$, the natural isomorphisms $\mathbf{1} \xrightarrow{\sim} \delta^! p_i^!$, $i = 1, 2$, give us maps

$$\mu_i\colon \delta_* \longrightarrow p_i^! \quad (i = 1, 2)$$

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as well as the base change isomorphism:

$$\beta = \beta(\clubsuit)\colon p_2^* f^! \xrightarrow{\ \sim\ } p_1^! f^*.$$

Since p_1 is of finite tor-dimension (it is flat!), β is an isomorphism between functors from $\mathbf{D}_{\mathrm{qc}}(Y)$ to $\mathbf{D}_{\mathrm{qc}}(X)$. $\leftarrow$ unbounded derived categories.

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Definition

The *fundamental class of f*

$$\mathrm{C}_f : \mathbf{L}\delta^*\delta_* f^* \to f^!,$$

a map between functors from $\mathbf{D}_{\mathrm{qc}}(Y)$ to $\mathbf{D}_{\mathrm{qc}}(X)$, is the composite

$$\mathbf{L}\delta^*\delta_* f^* \xrightarrow[\text{via } \mu_1]{} \mathbf{L}\delta^* p_1^! f^* \xrightarrow[\mathbf{L}\delta^*\beta^{-1}]{\sim} \mathbf{L}\delta^* p_2^* f^! \xrightarrow[\text{natural}]{\sim} f^!$$

This gives a canonical composite

$$\Omega_f^d[d] \longrightarrow H^{-d}(\mathbf{L}\delta^*\delta_*\mathscr{O}_X)[d] \longrightarrow H^{-d}(f^!\mathscr{O}_Y)[d] \longrightarrow f^!\mathscr{O}_Y$$

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whence a map (also called the *fundamental class*)

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On the other hand if f is *smooth* we have

$$(*) \qquad f^!\mathscr{O}_Y \xrightarrow{\sim} \mathbf{L}\delta^* p_2^* f^!\mathscr{O}_Y \underset{\beta}{\xrightarrow{\sim}} \mathbf{L}\delta^* p_1^! f^*\mathscr{O}_Y = \mathbf{L}\delta^* p_1^! \mathscr{O}_X$$

Since δ is a regular immersion, $\delta^! \xrightarrow{\sim} \mathbf{L}\delta^*(-) \otimes \wedge^d N_\delta[-d]$ ($N_\delta=$ normal bundle for δ) whence

$$(**) \qquad \mathbf{L}\delta^* \xrightarrow{\sim} \delta^!(-) \otimes_X \wedge^d N_\delta^*[d] = \delta^!(-) \otimes \Omega_f^d[d].$$

Note δ is clearly quasi-perfect (δ_* sends perfects to perfects).

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Note δ is clearly quasi-perfect (δ_* sends perfects to perfects). Substituting $(**)$ in $(*)$,i.e., in $f^! \mathscr{O}_Y \xrightarrow{\sim} \mathbf{L}\delta^* p_1^! \mathscr{O}_X$, we get Verdier's isomorphism:

$$v_f : f^! \mathscr{O}_Y \xrightarrow{\sim} \Omega_f^d[d].$$

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Theorem (Lipman-Neeman)

Let f be smooth of relative dimension d. Then

$$c_f = v_f^{-1}.$$

- $c_f \colon \Omega_f^d[d] \to f^!\mathscr{O}_Y$ is compatible with flat base change on Y. (Alonso-Jeremías-Lipman 2014).
- If f is *Cohen-Macaulay*, $f^!\mathscr{O}_Y$ is compatible with *arbitrary* base change, (and if f is smooth, so is Ω_f^d). (Lipman, 1979)
- In the smooth case, v_f is compatible with arbitrary base change (Sastry, 2004).

Lipman and Neeman then deduce that c_f is therefore compatible with arbitrary base change.

Below $\omega_f = \mathrm{H}^{-d}(f^!\mathscr{O}_Y)$ and we make the identification $f^!\mathscr{O}_Y = \omega_f[d]$. (Similarly for g.)

$$\begin{array}{ccc} U & \xrightarrow{\;u\;} & X \\ {\scriptstyle g}\downarrow & \clubsuit & \downarrow{\scriptstyle f} \\ V & \xrightarrow[\;v\;]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} u^*\Omega_f^d & \xrightarrow{\;u^*v_f\;} & u^*\omega_f \\ \Big\| & & \downarrow{\scriptstyle \beta(\clubsuit)} \\ \Omega_g^d & \xrightarrow[\;v_g\;]{} & \omega_g \end{array}$$

Difficulty with Verdier as a starting point

Suppose $f\colon X \to Y$ is smooth and proper of relative dimension d. Let $\int_f$ – *the Verdier trace/integral* – be defined by the commutativity of

$$
\begin{array}{ccc}
\mathrm{R}^d f_* \Omega_f^d & \xrightarrow[\mathrm{H}^0(v_f)]{\sim} & \mathrm{H}^0(\mathbf{R}f_* f^! \mathscr{O}_Y) \\
 & \searrow^{\int_f} & \downarrow{\scriptstyle \mathrm{H}^0(\mathrm{Tr}_f)} \\
 & & \mathscr{O}_Y
\end{array}
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We do not know $\int_f$, not even when $X = \mathbb{P}^d_Y$ and $f\colon X \to Y$ the structural map $\pi = \pi_Y\colon \mathbb{P}^d_Y \to Y$ and $Y = \operatorname{Spec} k$.

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Residues and Traces via Verdier

In recent (as yet unpublished) work, Suresh Nayak and I show

- When $f = \pi_Y$, $\int_f$ is the usual map $\mathrm{R}^d\pi_*\Omega^d_\pi \to \mathscr{O}_Y$.
- In other cases, $\int_f$ is determined through residues.

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$$\begin{array}{ccc} \mathrm{H}^d_Z(X,\, \Omega^d_f) & \twoheadrightarrow & \mathrm{H}^d(X,\, \Omega^d_f) \\ & \searrow^{\mathrm{res}_Z} & \downarrow{\scriptstyle \int_f} \\ & & A \end{array}$$

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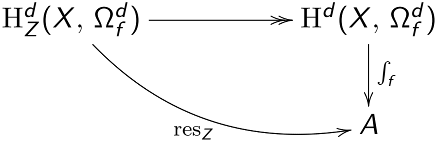
In the picture above $Y = \operatorname{Spec} A$. The *residue along Z*, $\mathrm{res}_Z = \mathrm{res}_{Z,f}$, is the composite indicated. $Z \to Y$ is finite dominant.



"Residues determine $\int$"

- If Z is contained in an affine open subscheme $U = \operatorname{Spec} R$ of Y, and is given up to radical by the vanishing of $t_1, \ldots, t_d$, then elements of $\mathrm{H}^d_Z(X, \Omega^d_f)$ can be represented by generalised fractions of the form $\left[\begin{smallmatrix} \mu \\ t_1^{\alpha_1}, \ldots, t_d^{\alpha_d} \end{smallmatrix}\right]$, with $\mu \in \Omega^d_{R/A}$.
- We show that the expressions

$$\mathrm{res}_{X/Y} \begin{bmatrix} \mu \\ t_1^{\alpha_1}, \ldots, t_d^{\alpha_d} \end{bmatrix} := \mathrm{res}_Z \begin{bmatrix} \mu \\ t_1^{\alpha_1}, \ldots, t_d^{\alpha_d} \end{bmatrix}$$

satisfy most of the residue formulae given in Hartshorne's [RD].

This is closely related to:
Neeman: *Traces and Residues*, Indiana U. Math. J., vol. 64, no. 1, 2015.

Thank you!

Thank you!

Happy Birthday, Amnon!

Verdier

For Verdier it's a different story. I remember Grothendieck had a great admiration for Verdier. He admired what we now call the Lefschetz-Verdier trace formula and Verdier's idea of defining $f^!$ first as a formal adjoint, and then calculating it later.

Bloch: I thought, maybe, that was Deligne's idea.

Illusie: No, it was Verdier's. But Deligne in the context of coherent sheaves used this idea afterward. Deligne was happy to somehow kill three hundred pages of Hartshorne's seminar in eighteen pages. (laughter)

Reminiscences of Grothendieck and his school. Luc Illusie with Alexander Beilinson, Spencer Bloch, Vladimir Drinfeld, et.al. Notices of the AMS, vol 57, no.9, Oct 2010.

" ... The abstract approach of Deligne and Verdier, and the more recent one of Neeman, seem on the surface to avoid many of the grubby details; but when you go beneath the surface to work out the concrete interpretations of the abstractly defined dualizing functors, it turns out to be not much shorter. I don't know of any royal road ... "

.

J. Lipman