

# LEFT ADJOINTS TO DIRECT IMAGES

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## 1. The topological case

For a topological space  $Z$ ,  $\mathcal{P}_Z$  will denote the category of presheaves on  $Z$  and  $\mathcal{S}h_Z$  the category of sheaves of abelian groups on  $Z$ .

**1.1. Inverse Image.** Let  $f: X \rightarrow Y$  be a continuous map between topological spaces. The inverse image  $f^{-1}\mathcal{G}$  of a sheaf  $\mathcal{G}$  on  $Y$  is the sheaf on  $X$  which is obtained by sheafifying the presheaf

$$(1.1.1) \quad U \mapsto \varinjlim_W \mathcal{G}(W).$$

where the direct limit varies over open sets  $W$  of  $Y$  such that  $U \subset f^{-1}(W)$ . The restriction maps are the obvious ones. In this section we denote the presheaf defined by (1.1.1) by  $f^*\mathcal{G}$ . The symbol  $f^\#\mathcal{G}$  will be used for a different functor in the next section. By the universal property of sheafification, for each such  $\mathcal{G}$  and each  $\mathcal{F} \in \mathcal{S}h_X$ , we have an isomorphism

$$(1.1.2) \quad \text{Hom}_{\mathcal{P}_X}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \text{Hom}_{\mathcal{S}h_X}(f^{-1}\mathcal{G}, \mathcal{F})$$

which is bifunctorial in  $\mathcal{F}$  and  $\mathcal{G}$  and such that when  $\mathcal{F} = f^{-1}\mathcal{G}$  the sheafification map  $f^*\mathcal{G} \rightarrow f^{-1}\mathcal{G}$  (which is an element in the group on the left side of (1.1.2)) maps to the identity element in  $\text{Hom}_{\mathcal{S}h_X}(f^{-1}\mathcal{G}, f^{-1}\mathcal{G})$  under (1.1.2).

**1.2. Adjointness.** For  $\mathcal{F} \in \mathcal{S}h_Y$  and  $\mathcal{G} \in \mathcal{S}h_X$ , we claim that there is a bifunctorial isomorphism

$$(1.2.1) \quad \text{Hom}_{\mathcal{S}h_X}(f^{-1}\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \text{Hom}_{\mathcal{S}h_Y}(\mathcal{G}, f_*\mathcal{F}).$$

The map (1.2.1) is defined as follows. If  $\varphi: f^{-1}\mathcal{G} \rightarrow \mathcal{F}$  is a map of sheaves, then, composing with the sheafification map  $f^*\mathcal{G} \rightarrow f^{-1}\mathcal{G}$ , we get a map of presheaves  $\bar{\varphi}: f^*\mathcal{G} \rightarrow \mathcal{F}$ . In fact, as a little thought shows, the map  $\varphi \mapsto \bar{\varphi}$  is the inverse of the isomorphism (1.1.2). Let  $W$  be an open subset of  $Y$ . Then we have a map  $\mathcal{G}(W) \rightarrow (f^*\mathcal{G})(f^{-1}(W))$  by the definition of direct limits. Consider the composite

$$\mathcal{G}(W) \longrightarrow (f^*\mathcal{G})(f^{-1}(W)) \xrightarrow{\bar{\varphi}(f^{-1}(W))} \mathcal{F}(f^{-1}(W)) = f_*\mathcal{F}(W).$$

It is easy to see this varies well with  $W$ , i.e., that as  $W$  varies, we get a map of sheaves  $\mathcal{G} \rightarrow f_*\mathcal{F}$ . We thus have a map

$$(\dagger) \quad \text{Hom}_{\mathcal{S}h_X}(f^{-1}\mathcal{G}, \mathcal{F}) \rightarrow \text{Hom}_{\mathcal{S}h_Y}(\mathcal{G}, f_*\mathcal{F}).$$

To show it is an isomorphism, suppose we have  $\psi \in \text{Hom}_{\mathcal{S}h_Y}(\mathcal{G}, f_*\mathcal{F})$ . Let  $U$  be an open set in  $X$  and  $W$  an open set in  $Y$  such that  $f^{-1}(W) \supset U$ . We have a map

$$(*)_W \quad \mathcal{G}(W) \longrightarrow \mathcal{F}(U)$$

given by the composite

$$\mathcal{G}(W) \xrightarrow{\psi(W)} f_* \mathcal{F}(W) = \mathcal{F}(f^{-1}(W)) \longrightarrow \mathcal{F}(U).$$

One checks that if  $W'$  is an open subset of  $Y$  such that  $f(U) \subset W' \subset W$  then the diagram

$$\begin{array}{ccc} \mathcal{G}(W) & & \\ \text{restriction} \downarrow & \searrow^{(*)_W} & \\ \mathcal{G}(W') & \xrightarrow{(*)_{W'}} & \mathcal{F}(U) \end{array}$$

commutes. Taking direct limits we get a map

$$\tilde{\psi}(U): f^* \mathcal{G}(U) \rightarrow \mathcal{F}(U)$$

and for  $V$  an open subset of  $X$  contained in  $U$ , one checks from the definitions that the following diagram commutes

$$\begin{array}{ccc} f^* \mathcal{G}(U) & \xrightarrow{\tilde{\psi}(U)} & \mathcal{F}(U) \\ \text{restriction} \downarrow & & \downarrow \text{restriction} \\ f^* \mathcal{G}(V) & \xrightarrow{\tilde{\psi}(V)} & \mathcal{F}(V), \end{array}$$

i.e.,  $\tilde{\psi}: f^* \mathcal{G} \rightarrow \mathcal{F}$  is a map of presheaves. We thus have a map

$$\text{Hom}_{\text{Sh}_Y}(\mathcal{G}, f_* \mathcal{F}) \longrightarrow \text{Hom}_{\text{P}_X}(f^* \mathcal{G}, \mathcal{F})$$

given by  $\psi \mapsto \tilde{\psi}$ . Applying (1.1.2) we therefore get

$$(\ddagger). \quad \text{Hom}_{\text{Sh}_Y}(\mathcal{G}, f_* \mathcal{F}) \longrightarrow \text{Hom}_{\text{Sh}_X}(f^{-1} \mathcal{G}, \mathcal{F})$$

It is easy to see that  $(\dagger)$  and  $(\ddagger)$  are inverses of each other.

## 2. The case of schemes

**2.1.** Now suppose  $f: X \rightarrow Y$  is a map of schemes. We claim that for an  $\mathcal{O}_X$ -module  $\mathcal{F}$  and an  $\mathcal{O}_Y$ -module  $\mathcal{G}$ , we have a bifunctorial isomorphism

$$(2.1.1) \quad \text{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F}).$$

We now change notation and write  $f^* \mathcal{G}$  for the presheaf

$$U \mapsto f^{-1} \mathcal{G}(U) \otimes_{f^{-1} \mathcal{O}_Y(U)} \mathcal{O}_X(U).$$

By definition  $f^* \mathcal{G}$  is the sheafification of  $f^* \mathcal{G}$ . Note that  $f^* \mathcal{G}$  is an  $\mathcal{O}_X$ -module. Recall that if  $A \rightarrow B$  is a map of (commutative) rings, then

$$(*) \quad \text{Hom}_B(M \otimes_A B, N) \xrightarrow{\sim} \text{Hom}_A(M, N).$$

Now suppose we have an  $\mathcal{O}_X$ -module map  $f^* \mathcal{G} \rightarrow \mathcal{F}$ . For each open  $U$  in  $X$  we have a map  $f^{-1} \mathcal{G}(U) \otimes_{f^{-1} \mathcal{O}_Y(U)} \mathcal{O}_X(U) \rightarrow \mathcal{F}(U)$ , and hence (via  $(*)$ ) a map  $f^{-1} \mathcal{G}(U) \rightarrow \mathcal{F}(U)$ . The maps  $f^{-1} \mathcal{G}(U) \rightarrow \mathcal{F}(U)$  behave well with respect to the open sets  $U$  giving us a map of sheaves  $f^{-1} \mathcal{G} \rightarrow \mathcal{F}$ . (This map is easily seen to be a map of  $f^{-1} \mathcal{O}_Y$  modules.) From the first section we know that we therefore have

a map of sheaves  $\mathcal{G} \rightarrow f_*\mathcal{F}$ . It is not hard to see that this is a map of  $\mathcal{O}_Y$ -modules, since it arises from a map of  $f^{-1}\mathcal{O}_Y$ -modules. We thus have a map:

$$(**) \quad \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F}).$$

Conversely, given a map of  $\mathcal{O}_Y$ -modules  $\psi: \mathcal{G} \rightarrow f_*\mathcal{F}$ , we get from the first section a map  $\hat{\psi}: f^{-1}\mathcal{G} \rightarrow \mathcal{F}$  and this is seen easily to be a  $f^{-1}\mathcal{O}_Y$ -module map. From (\*), we have the identification

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_Y(U)}(f^{-1}\mathcal{G}(U), \mathcal{F}(U)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X(U)}(f^{-1}\mathcal{G}(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U), \mathcal{F}(U)).$$

Thus the  $f^{-1}\mathcal{O}_Y$ -module map  $\hat{\psi}: f^{-1}\mathcal{G} \rightarrow \mathcal{F}$  gives rise to a  $\mathcal{O}_X$ -module map  $f^*\mathcal{G} \rightarrow \mathcal{F}$ . In other words (\*\*) is an isomorphism, since the two processes are easily seen to be inverses of each other. Moreover, from the universal property of sheafification, we get an  $\mathcal{O}_X$ -module map  $f^*\mathcal{G} \rightarrow \mathcal{F}$ . This essentially establishes (2.1.1). For good book-keeping, here is the hierarchy of identifications:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, \mathcal{F}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

The first isomorphism above is from the universal property of sheafification (which one checks respects  $\mathcal{O}_X$ -structures). The second is (\*\*) which we have shown is an isomorphism. The third is from the topological case considered in section one, and noting that the adjunction isomorphism there is such that if  $f^{-1}\mathcal{G} \rightarrow \mathcal{F}$  is a map of  $f^{-1}\mathcal{O}_Y$ -modules, then its adjoint, namely  $\mathcal{G} \rightarrow f_*\mathcal{F}$ , is a map of  $\mathcal{O}_Y$ -modules.

#### REFERENCES

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- [H] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.