SPECTRAL SEQUENCES FOR BIGRADED COMPLEXES

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There are very few proofs here. I am outlining a utilitarian approach to spectral sequences concentrating on those facts which are useful to algebraic geometers. The idea is to take the properties (1)–(4) listed in Subsection 1.2 and Theorem 1.2.3 as black-boxes and see what can be deduced. A considerable amount can be, using elementary arguments. All the "non-elementariness" is pushed to the black-boxes mentioned. In my experience, proving the "black-box statements" is not difficult either. But they seem artificial statements the first time one sees them (and the second time, and the third time, . . .). It might be an idea to see what one can do with them. This is all the spectral sequences I have ever used in research, and I often forget what happens when $E_2^{p,q}$'s vanish for certain (p, q)'s. (Do we have a map from the total cohomology to $E_r^{p,q}$ or is it the other way round? And is the map surjective, injective? And when is it an isomorphism?) I have always successfully deduced what is true from those black-boxes. I should add that I always remember regions R_1, R_2, R_3, and R_4, but never really remember what vanishing on each implies. I just deduce it every time (doesn't take long). In any case, with derived categories, who really needs spectral sequences?

Throughout $K^\bullet$ will be a complex of modules over a ring A with coboundary ∂. Our notations and assumptions are as follows.

(1) The underlying graded module of $K^\bullet$ has a bigrading:

$$K^n = \bigoplus_{p+q=n} K^{p,q}.$$

(2) We will write

$$\partial_r^{p,q} \colon K^{p,q} \to K^{p+r,q-r+1}$$

for the composite $K^{p,q} \hookrightarrow K^{p+q} \xrightarrow{\partial^{p+q}} K^{p+q+1} \twoheadrightarrow K^{p+r,q-r+1}$.

(3) We will assume that

$$K^{p,q} = 0 \quad (p < 0 \text{ or } q < 0)$$

and further that

$$\partial_r^{p,q} = 0 \quad (r < 0).$$

1. Basics

1.1. **Filtrations.** Let $F^iK^\bullet$ be the graded sub-module of $K^\bullet$ whose n-th graded piece is

$$F^iK^n = \bigoplus_{p+q=n,\, p\geq i} K^{p,q}.$$

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The assumption that $\partial^{p,q} = 0$ for $p < 0$ implies that $\partial^n(F^iK^n) \subset F^iK^{n+1}$. Thus $F^iK^\bullet$ is a subcomplex of $K^\bullet$ for every in integer i, and we have a decreasing filtration of complexes

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset \cdots \supset F^nK^\bullet \supset \ldots .$$

For each n and each p we have a map $\mathrm{H}^n(F^pK^\bullet) \to \mathrm{H}^n(K^\bullet)$.[1] Let the image of this map be denoted $F^p\mathrm{H}^n(K^\bullet)$. Then we have a another filtration

$$\mathrm{H}^n(K^\bullet) = F^0\mathrm{H}^n(K^\bullet) \supset F^1\mathrm{H}^n(K^\bullet) \supset \cdots \supset F^n\mathrm{H}^n(K^\bullet) \supset \ldots .$$

The idea is to compute $\mathrm{H}^n(K^\bullet)$ via $F^p\mathrm{H}^n(K^\bullet)$ and to compute the latter using $\mathrm{H}^n(F^pK^\bullet)$ and the surjective map $\mathrm{H}^n(F^pK^\bullet) \twoheadrightarrow F^p\mathrm{H}^n(K^\bullet)$. Unfortunately it is not so simple to compute $\mathrm{H}^n(F^pK^\bullet)$, or to work out the comparison map $\mathrm{H}^n(F^pK^\bullet) \twoheadrightarrow F^p\mathrm{H}^n(K^\bullet)$ explicitly. One uses approximations and "in the limit" one obtains only the graded pieces $F^p\mathrm{H}^n(K^\bullet)/F^{p+1}\mathrm{H}^n(K^\bullet)$. If there is extra information available, one can reconstruct $\mathrm{H}^n(K^\bullet)$ from the graded pieces for the filtration $\{F^p\mathrm{H}(K^\bullet)\}$, i.e., the modules $F^p\mathrm{H}^n(K^\bullet)/F^{p+1}\mathrm{H}^n(K^\bullet)$.

1.2. **The modules $E_r^{i,j}$.** To lighten notation let $H^n = \mathrm{H}^n(K^\bullet)$ and $\mathrm{Gr}^pH^n = F^pH^n/F^{p+1}H^n$. Here is how Gr^pH^n is obtained as a "limit". First, for each triple of integers (i, j, r) there are modules $E_r^{i,j}$, which we will define later (see (1.2.5)), and maps

$$d_r\colon E_r^{i,j} \to E_r^{i+r,j-r+1}$$

such that $d_r^2 = 0$ (see (1.2.6)). These have the following properties.

(1) For a pair of integers (p, q),

$$E_0^{p,q} = K^{p,q}$$

and

$$d_0 = \partial_0^{p,q}.$$

(2) Consider the short exact sequence of complexes

$$0 \longrightarrow F^{p+1}K^\bullet \longrightarrow F^pK^\bullet \longrightarrow F^pK^\bullet/F^{p+1}K^\bullet \longrightarrow 0.$$

Then

$$E_1^{p,q} = \mathrm{H}^q(F^pK^\bullet/F^{p+1}K^\bullet).$$

The map $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$ is the one induced by the map $\partial_1^{p,q}\colon F^pK^\bullet \to F^{p+1}K^\bullet$.

(3) For each triple of integers (i, j, r), the module $E_{r+1}^{i,j}$ is given by the formula

$$\frac{\ker\,(E_r^{i,j} \xrightarrow{d_r} E_r^{i+r,j-r+1})}{\mathrm{im}\,(E_r^{i-r,j+r-1} \xrightarrow{d_r} E_r^{i,j})} = E_{r+1}^{i,j}. \tag{1.2.1}$$

(4) Since $E_r^{i,j}$ is a sub-quotient of $E_{r-1}^{i,j}$, therefore it is a subquotient of $K^{i,j} = E_0^{i,j}$. Thus the module $E_r^{i,j} = 0$ if either i or j is negative. In particular, if $r > j+1$, then the map $d_r\colon E^{i,j} \to E^{i+r,j-r+1}$ is zero. This means that for $r > j+1$ we have a surjective map, $E_r^{i,j} \twoheadrightarrow E_{r+1}^{i,j}$. Composing these we get surjective maps

$$\theta_s^r\colon E_r^{i,j} \twoheadrightarrow E_s^{i,j}, \qquad (j+1 < r < s). \tag{1.2.2}$$

[1]Note that for $i \le 0$, $F^iK^\bullet = K^\bullet$ and $F^i\mathrm{H}^n(K^\bullet) = \mathrm{H}^n(K^\bullet)$, and in this case the map is the identity.

Thus for fixed (i,j), the collection $\{E_r^{i,j}\}_{r|r>j+1}$ forms a directed system.

The main result is:

Theorem 1.2.3. *With the above notations we have*

$$\mathrm{Gr}^p H^{p+q} = \varinjlim_r E_r^{p,q}$$

where for fixed (p,q), $\{E_r^{p,q}\}_r$ *is the direct system arising from the maps* (1.2.2).

Here is the official definition of $E_r^{i,j}$ for a triple (i,j,r) of integers. First, define the following modules:

$$Z_r^{i,j} = \{x \mid x \in F^i K^{i+j} \text{ and } dx \in F^{i+r} K^{i+j+1}\} \tag{1.2.4}$$
$$B_r^{i,j} = \{x \mid x \in F^i K^{i+j} \text{ and } x = dy \text{ for some } y \in F^{p-r} K^{i+j-1}\}.$$

Then we define $E_r^{i,j}$ in the following way:

$$E_r^{i,j} = Z_r^{i,j} / (B_{r-1}^{i,j} + Z_{r-1}^{i+1,j-1}). \tag{1.2.5}$$

Note that $\partial(Z_r^{i,j}) \subset Z^{i+r,j-r+1}$ whence we have a map $Z^{i,j} \to E_r^{i+r,j-r+1}$. It is easy to check that this map vanishes on $B_{r-1}^{i,j} + Z_{r-1}^{i+1,j-1}$, and hence we get the map

$$d_r \colon E_r^{i,j} \to E_r^{i+r,j-r+1}. \tag{1.2.6}$$

Remark 1.2.7. The direct system $\{E_r^{p,q}\}_{r|r>q+1}$ stabilises in a finite number of steps. Indeed we have already seen that $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ is zero if $r > q+1$. Similarly, using the fact that $E_r^{i,j} = 0$ if $i < 0$ (see property (4) in the itemised list above), we see that $E^{p-r,q+r-1} = 0$ if $r > p$. Thus

$$E_r^{p,q} = E_s^{p,q} \qquad (\max\{p, q+1\} < r \leq s).$$

In particular, via Theorem 1.2.3, we get

$$Gr^p H^{p+q} = F^p H^{p+q} / F^{p+1} H^{p+q} = E_r^{p,q} \qquad (\max\{p, q+1\} < r).$$

1.3. **Terminology and Notations.** The collection $\{E_r^{p,q}\}_{p,q,r}$ together with the collection of maps $\{d_r\}$ is the *spectral sequence* associated with the bigraded complex $K^\bullet$. The relationship between the $E_r^{p,q}$ and the cohomologies $\mathrm{H}^n(K^\bullet)$ is described by the phrase $E_r^{p,q}$ *abuts to* $\mathrm{H}^{p+q}(K^\bullet)$. The following shorthand is often used to denote this "abutment"

$$E_r^{p,q} \Longrightarrow_p \mathrm{H}^n(K^\bullet)$$

where it is understood that $n = p+q$. Since $E_2^{p,q}$ is often the most useful term of the spectral sequence, one also writes

$$E_2^{p,q} \Longrightarrow \mathrm{H}^{p+q}(K^\bullet)$$

as a shorthand for $E_r^{p,q}$ *abuts to* $\mathrm{H}^n(K^\bullet)$.

2. Getting information about $\mathrm{H}^n(K^\bullet)$ from $E_r^{p,q}$

In what follows we fix a point (p,q) in $\mathbb{R}^2$ such that p and q are positive integers and set $n = p+q$. Since $\mathrm{Gr}^p H^n = \varinjlim_r E_r^{p,q}$ we will often write

$$E_\infty^{p,q} = \mathrm{Gr}^p H^n.$$

We will assume $r \geq 2$, though in most applications I know $r = 2$.[2]

[2] In the Hodge to DeRham spectral sequence, the case $r = 1$ is the critical one.

2.1. **Regions in the grid.** Define the following four subsets of $\mathbb{R}^2$:

$$\begin{aligned}
R_1 &= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n, y>q\}\\
&= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n, x<p\},\\
R_2 &= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n, y<q\}\\
&= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n, x>p\},\\
R_3 &= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n-1, y>q\}\\
&= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n-1, x<p-1\},\\
R_4 &= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n+1, y<q\}\\
&= \{(x,y)\in\mathbb{Z}^2 \mid x+y=n+1, x>p+1\}.
\end{aligned}$$

Interesting things can be said about $\mathrm{H}^n(K^\bullet)$ when $\{E_r^{p,q}\}$ vanish along these regions. We take this up in the next few subsections.

Remark 2.1.1. The regions R_i depend upon (p,q). So perhaps one should have indexed them as $R_i(p,q)$.

2.2. $E_r^{i,j}$ **vanishing on** R_k**.** Here are the elementary arguments on which things hinge.

2.2.1. *Vanishing on* R_1. Suppose $E_r^{i,j}=0$ for all $(i,j)\in R_1$. Since every $E_s^{i,j}$, $s\geq r$, is a sub-quotient of $E_r^{i,j}$, we have $E_s^{i,j}=0$ for $(i,j)\in R_1$ and $s\geq r$. Thus, going to the direct limit,

$$\mathrm{Gr}^i H^n = 0 \qquad (i=0,\dots,p-1).$$

This means

$$H^n = F^0H^n = F^1H^n = \cdots = F^pH^n$$

and hence

$$\mathrm{Gr}^pH^n = H^n/F^{p+1}H^n.$$

Thus we have a surjective map

$$H^n \twoheadrightarrow E_\infty^{p,q}.$$

2.2.2. *Vanishing on* R_2. Suppose $E_r^{i,j}=0$ for $(i,j)\in R_2$. Then arguing as we did above, $\mathrm{Gr}^iH^n=0$ for $i>p$. This means

$$F^{p+1}H^n = F^{p+2}H^n = \cdots = F^{n+1}H^n = 0.$$

(The last equality follows from the relation $F^{n+1}K^n=0$.) Thus $\mathrm{Gr}^pH^n = F^pH^n$. In other words we have an injective map

$$E_\infty^{p,q} = F^pH^n \hookrightarrow H^n.$$

2.2.3. *Vanishing on* R_3. If $E_r^{i,j}=0$ for $(i,j)\in R_3$ then $E_s^{i,j}=0$ for all $s\geq r$ and $(i,j)\in R_3$. For such s, we have $E_s^{p-s,q+s-1}=0$ since $(p-s,q+s-1)\in R_3$ (recall $s\geq r\geq 2$). It follows that $E_{s+1}^{p,q}=\ker(E_s^{p,q}\xrightarrow{d_s}E_s^{p+s,q-s+1})$. Thus we have injective maps

$$\ldots \hookrightarrow E_s^{p,q} \hookrightarrow \ldots E_{r+2}^{p,q} \hookrightarrow E_{r+1}^{p,q} \hookrightarrow E_r^{p,q}.$$

In particular we have an injective map

$$E_\infty^{p,q} \hookrightarrow E_r^{p,q}.$$

2.2.4. *Vanishing on R_4.* If $E_r^{i,j} = 0$ for $(i,j) \in R_4$ then $\ker(E_s^{p,q} \xrightarrow{d_s} E^{p+s,q-s+1}{}_s) = E_s^{p,q}$ for all $s \geq r$. Thus we have surjective maps

$$E_r^{p,q} \twoheadrightarrow E_{r+1}^{p,q} \twoheadrightarrow E_{r+2}^{p,q} \twoheadrightarrow \cdots \twoheadrightarrow E_s^{p,q} \quad (r \leq s).$$

In particular we have a surjective map

$$E_r^{p,q} \twoheadrightarrow E_\infty^{p,q}.$$

Remarks 2.2.1. 1) If $E_r^{i,j} = 0$ on $R_1 \cup R_2$ then $Gr^i H^n = 0$ for $i \neq p$, whence $H^n = F^0 H^n = F^1 H^n = \cdots = F^p H^n$ and $F^{p+1} H^n = F^{p+2} H^n = \cdots = F^{n+1} H^n = 0$. Thus $H^n = F^p H^n = E_\infty^{p,q}$.

2) Similarly if $E_r^{i,j} = 0$ on $R_3 \cup R_4$ we have an isomorphism

$$E_r^{p,q} \xrightarrow{\sim} E_\infty^{p,q}.$$

Rather than appealing to the edge homomorphisms, one notes that $E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_{r+j}^{p,q} = \cdots = E_\infty^{p,q}$ for $j \geq 0$.

The useful result is the following (proofs from arguments given above the Remarks and in the Remarks):

Proposition 2.2.2. *With notations as above, we have:*

(a) *If $E_r^{i,j} = 0$ for $(i,j) \in R_1 \cup R_3$ then there is a map*

$$\mathrm{H}^n(K^\bullet) \to E_r^{p,q}$$

which factors as

$$\mathrm{H}^n(K^\bullet) \twoheadrightarrow E_\infty^{p,q} \hookrightarrow E_r^{p,q}.$$

(b) *If $E_r^{i,j} = 0$ for $(i,j) \in R_2 \cup R_4$ then there is a map*

$$E_r^{p,q} \to \mathrm{H}^n(K^\bullet)$$

which factors as

$$E_r^{p,q} \twoheadrightarrow E_\infty^{p,q} \hookrightarrow \mathrm{H}^n(K^\bullet).$$

The maps in (a) and (b) are called edge homomorphisms*.*

(c) *If $E_r^{i,j} = 0$ for $(i,j) \in R_1 \cup R_2$ then*

$$\mathrm{H}^n(K^\bullet) = E_\infty^{p,q}.$$

(d) *If $E_r^{i,j} = 0$ for $(i,j) \in R_3 \cup R_4$ then*

$$E_r^{p,q} \xrightarrow{\sim} E_\infty^{p,q}.$$

(e) *If $E_r^{i,j} = 0$ for $(i,j) \in R_1 \cup R_2 \cup R_3$ then the edge homomorphism from (a) is injective, i.e., we have an injection*

$$\mathrm{H}^n(K^\bullet) \hookrightarrow E_r^{p,q}.$$

(f) *If $E_r^{i,j} = 0$ for $(i,j) \in R_1 \cup R_2 \cup R_4$ then the edge homomorphism in (b) is surjective, i.e., we have a surjective map*

$$E_r^{p,q} \twoheadrightarrow \mathrm{H}^n(K^\bullet).$$

(g) *If $E_r^{i,j} = 0$ for $(i,j) \in R_1 \cup R_2 \cup R_3 \cup R_4$ then we have an isomorphism*

$$E_r^{p,q} \xrightarrow{\sim} \mathrm{H}^n(K^\bullet).$$

Remark 2.2.3. The requirement that $K^{\bullet,\bullet}$ live in the "first quadrant" can clearly be relaxed in the above considerations. The above results hold if there exist integers p_0 and q_0 such that $K^{p,q} = 0$ is either $p < p_0$ or $q < q_0$. Moreover the requirement that $K^{p,q}$ be A-modules can also be relaxed. We can work over an abelian category $\mathscr{A}$ and the results hold true.

3. Examples

The following two examples (the first concerning dominant maps of varieties and the second concerning proper maps between noetherian schemes) are to show how one uses Proposition 2.2.2 if one accepts the form of $E_2^{p,q}$ associated with the Leray spectral sequence. Nothing more than the E_2 term of this spectral sequence is needed. The spectral sequence itself is described in a later section of these notes for completeness. That should not hinder understanding of what is given in this section.

3.1. **Dominant map of varieties.** This example is essentially [L1, pp.42–43, Remark (4.3.1)]. Let k be a field, $f\colon V \to W$ a map between *proper* k-varieties[3] such that f is surjective, $\dim W = r$, $\dim V = r + d$. Let $\mathscr{G}$ be a quasi-coherent $\mathscr{O}_V$-module. We have a spectral sequence, the *Leray spectral sequence* (see Subsection 4.3 below for details on its definition)

$$\mathrm{H}^p(W,\, \mathrm{R}^q f_*\mathscr{G}) \Longrightarrow \mathrm{H}^{p+q}(V,\, \mathscr{G}).$$

Let $p = r$ and let $q \geq 0$ be arbitrary and consider the regions R_i, $i = 1, 2, 3, 4$ for this choice of (p, q). Since $\dim W = r$ therefore

$$\mathrm{H}^i(W,\, \mathrm{R}^j f_*\mathscr{G}) = 0 \qquad\qquad (i > r,\ j \geq 0).$$

This means that $E_2^{ij} = 0$ for $(i, j) \in R_2 \cup R_4$. Hence we have a canonical map

$$\mathrm{H}^r(W,\, \mathrm{R}^q f_*\mathscr{G}) \longrightarrow \mathrm{H}^{r+q}(V,\, \mathscr{G}).$$

The quasi-coherent sheaf $\mathrm{R}^q f_*\mathscr{G}$ is supported on a closed subset of W over which the fibres of f have dimension at least q. To see this, use first the fact that $\mathscr{G}$ is the direct limit of coherent sheaves, to reduce to the case where $\mathscr{G}$ is coherent, and then appeal to [EGA-III, (4.2.2)]. The statement about direct limits can be found in [EGA-01, p.320, (6.9.12)]. We point out that cohomologies of sheaves commute with direct limits. Coming to where we were before we got distracted, $\mathrm{R}^q f_*\mathscr{G}$ is supported on a closed subset of W over which the fibres of f have dimension at least q. If $q > d$ then this closed subset has dimension at most $r + d - q - 1$, whence

$$\mathrm{H}^i(W,\, \mathrm{R}^j f_*\mathscr{G}) = 0 \qquad\qquad (j > d,\ i + j > r + d).$$

Setting $(p, q) = (r, d)$ to work out the regions R_1, R_2, R_3, and R_4, we see that $E_2^{i,j} = 0$ for $(i,\, j) \in R_1 \cup R_2 \cup R_4$. Hence we have a surjective map

$$\mathrm{H}^r(W,\, \mathrm{R}^d f_*\mathscr{G}) \longrightarrow \mathrm{H}^{r+d}(V,\, \mathscr{G}).$$

If further f is flat, all the fibres are of dimension d and we have $\mathbf{R}^j f_*\mathscr{G} = 0$ for $j \geq d$, again by reducing to coherent $\mathscr{G}$ via a direct limit argument, and using semi-continuity for coherent $\mathscr{G}$. In this case the region R_1 for $(p, q) = (r, d)$ is also a region where $E_2^{i,j}$ vanishes, and hence we have an isomorphism

$$\mathrm{H}^r(W,\, \mathrm{R}^d f_*\mathscr{G}) \xrightarrow{\ \sim\ } \mathrm{H}^{r+d}(V,\, \mathscr{G}).$$

[3] A k-variety will mean a separated, finite-type k-scheme which is reduced and irreducible.

3.2. **Proper maps and coherence.** One of the nicest examples of the use of the Leray spectral sequence is the proper mapping theorem of Grothendieck. The theorem in the context of coherent analytic sheaves is due to Grauert. When f is a projective morphism, the result is due to Serre (at least for varieties over a field) and is given in Hartshorne's *Algebraic Geometry* (see [H, p.228, Theorem 5.2 (a)]). The general case follows from the special case, but one needs spectral sequences and Chow's Lemma as we shall see below.

Theorem 3.2.1. *Let $f\colon X \to Y$ be a proper map of schemes with Y noetherian. If $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, then the $\mathscr{O}_Y$-modules $\mathrm{R}^p f_*\mathscr{F}$, $p \geq 0$, are coherent.*

Proof. We will use noetherian induction on the support of $\mathscr{F}$. Recall that the *principle of noetherian induction* is the following:

Suppose Z is a noetherian scheme and $\mathbf{P}$ *a property of closed subschemes of Z. Suppose further that for every closed subscheme V of X the following implication holds:*

$$(\textit{Every strict closed subscheme of } V \textit{ has property } \mathbf{P}) \Longrightarrow (V \textit{ has property } \mathbf{P}).$$

Then every closed subscheme of Z has the property $\mathbf{P}$*.*

By a "strict closed subscheme" W of V, we mean W is a closed is a closed subscheme of V but $W \neq V$.

Since f is proper and Y is noetherian, X is noetherian and therefore the principle of noetherian induction is a valid principle. Thus we may assume $\mathrm{R}^p f_*\mathscr{G}$ is coherent for all $p \geq 0$ whenever $\mathscr{G}$ is a coherent $\mathscr{O}_X$-module such that $\operatorname{Supp}\mathscr{G} \subsetneq X$ (note that if $\mathscr{G}$ is coherent and supported at a closed point of X, then $\mathrm{R}^p f_*\mathscr{G}$ is coherent for all $p \geq 0$). To be very precise, we are regarding $\operatorname{Supp}\mathscr{G}$ as a closed subscheme of X, not merely as a closed subset of X, and the relationship "$\subsetneq$" is a statement about schemes and not just about the underlying sets. If we show $\mathrm{R}^p f_*\mathscr{F}$ is coherent under this induction hypothesis, we are done.

By Chow's Lemma (see, for example, [H, p.107, Exercise 4.10]) we can find a projective map $\pi\colon X' \to X$ such that $f \circ \pi$ is proper and such that there is a dense open subscheme U of X with the property that $\pi^{-1}(U) \to U$ is an isomorphism. Fix such a π. We have the Leray spectral sequence (see Subsection 4.3)

$$(\mathrm{R}^p f_* \circ \mathrm{R}^q \pi_*)(\pi^*\mathscr{F}) \Longrightarrow \mathrm{R}^{p+q}(f\circ\pi)_*(\pi^*\mathscr{F}).$$

For $j \geq 1$, $\operatorname{Supp}(\mathrm{R}^j\pi_*\pi^*\mathscr{F}) \subsetneq X$ since $\mathrm{R}^j\pi_*\pi^*\mathscr{F}$ is supported outside U. By our induction hypothesis this means that $(\mathrm{R}^p f_* \circ \mathrm{R}^j\pi_*)(\pi^*\mathscr{F})$ is coherent for $p \geq 0$ and $j \geq 1$. In other words $E_2^{p,j}$ is coherent for $p \geq 0$ and $j \geq 1$. We will now show that $E_2^{p,0}$ is also coherent for $p \geq 0$. Fix such a p. First note that the regions R_2 and R_4 for $(p,0)$ are in the loci for the vanishing of $E_2^{i,j}$. This means that we have a surjective map $E_2^{p,0} \twoheadrightarrow E_\infty^{p,0}$, and an injective map $E_\infty^{p,0} \hookrightarrow \mathrm{R}^p(f\circ\pi)_*(\mathscr{F})$. The map $f\circ\pi$ is projective, therefore $\mathrm{R}^p(f\circ\pi)_*(\mathscr{F})$ is coherent (cf. [H, p.228, Theorem 5.2 (a)]). It follows that $E_\infty^{p,0}$ is coherent. With J equal to the kernel of $E_2^{p,0} \to E_\infty^{p,0}$, we have a short exact sequence

$$0 \longrightarrow J \longrightarrow E_2^{p,0} \longrightarrow E_\infty^{p,0} \longrightarrow 0.$$

To show $E_2^{p,0}$ is coherent, it is enough to show that J is coherent, which we now proceed to do.

Since $E_r^{p,-r+1}=0$ for all $r\geq 2$, the map $E_r^{p,0}\to E_{r+1}^{p,0}$ is surjective for all $r\geq 2$. For $r\geq 2$ let $K_r=\ker(E_r^{p,0}\twoheadrightarrow E_{r+1}^{p,0})$ and $J_r=\ker(E_2^{p,0}\twoheadrightarrow E_r^{p,0})$. Note $J_2=0$ and $J_3=K_2$. Clearly $K_r\cong J_{r+1}/J_r$ for $r\geq 2$.

Now, $E_{p+j}^{-j,p+j-1}=0$ for $j\geq 1$. This means that $d_{p+j}\colon E_{p+j}^{-j,p+j-1}\to E_{p+j}^{p,0}$ is the zero map. Hence $K_r=0$ for $r\geq p+1$. In other words, we have

$$E_{p+1}^{p,0}=E_{p+2}^{p,0}=E_{p+3}^{p,0}=\cdots=E_\infty^{p,0}.$$

The surjective map $E_2^{p,0}\twoheadrightarrow E_\infty^{p,0}$ factors as

$$E_2^{p,0}\twoheadrightarrow E_3^{p,0}\twoheadrightarrow E_4^{p,0}\cdots\twoheadrightarrow E_r^{p,0}\cdots\twoheadrightarrow E_{p+1}^{p,0}=E_\infty^{p,0}.$$

We have a filtration.

$$0=J_2\subset J_3\subset J_4\subset\cdots\subset J_{p+1}=J.$$

For $j\geq 1$ and $r\geq 2$, $E_r^{i,j}$ is a subquotient of $E_2^{i,j}$, and hence is coherent. In particular, for $r\geq 2$, $E_r^{p-r,r-1}$ is coherent. Thus the image of $E_r^{p-r,r-1}$ under d_r, namely K_r, is coherent. In particular $J_3(=K_2)$ is coherent. Suppose J_r is coherent for some $r\geq 3$. Consider the short exact sequence

$$0\longrightarrow J_r\longrightarrow J_{r+1}\longrightarrow K_r\longrightarrow 0.$$

Since K_r and J_r are coherent, we conclude that J_{r+1} is coherent. Thus J_r is coherent for all $r\geq 3$. In particular $J=J_{p+1}$ is coherent. As we argued earlier, this means $\mathrm{R}^pf_*(\pi_*\pi^*\mathscr{F})$ is coherent.

Now consider the exact sequence of $\mathscr{O}_X$-modules:

$$0\longrightarrow\mathscr{K}\longrightarrow\mathscr{F}\longrightarrow\pi_*\pi^*(\mathscr{F})\longrightarrow\mathscr{C}\longrightarrow 0$$

where $\mathscr{K}$ and $\mathscr{C}$ are respectively the kernel and cokernel of $\mathscr{F}\to\pi_*\pi^*(\mathscr{F})$. Clearly $\mathscr{K}$ and $\mathscr{C}$ are supported outside U and hence by our induction hypothesis, $\mathrm{R}^pf_*\mathscr{K}$ and $\mathrm{R}^pf_*\mathscr{C}$ are coherent. Let $\mathscr{G}$ be the image of the map $\mathscr{F}\longrightarrow\pi_*\pi^*(\mathscr{F})$. The above exact sequence breaks up into the following two short exact sequences:

$$0\longrightarrow\mathscr{K}\longrightarrow\mathscr{F}\longrightarrow\mathscr{G}\longrightarrow 0 \tag{3.2.2}$$

and

$$0\longrightarrow\mathscr{G}\longrightarrow\pi_*\pi^*\mathscr{F}\longrightarrow\mathscr{C}\longrightarrow 0. \tag{3.2.3}$$

Since $\mathrm{R}^pf_*\mathscr{C}$ and $\mathrm{R}^pf_*(\pi_*\pi^*\mathscr{F})$ are coherent for all $p\geq 0$, the long exact sequence of higher direct images of f associated with (3.2.3) shows that $\mathrm{R}^pf_*\mathscr{G}$ is coherent for all $p\geq 0$. We repeat this argument for the long exact sequence arising from (3.2.2) (noting the fact that $\mathrm{R}^pf_*\mathscr{G}$ and $\mathrm{R}^pf_*\mathscr{K}$ are coherent for $p\geq 0$) to get the required result. □

4. Some Double Complexes and associated Spectral Sequences

4.1. **Cartan-Eilenberg resolutions.** Suppose $\mathscr{A}$ is an abelian category with enough injectives, and $C^\bullet$ a bounded below complex in $\mathscr{A}$, say $C^q=0$ if $q<q_0$. One can find a double-complex $I^{\bullet,\bullet}$ (whose total complex is written $I^\bullet$) of injectives in $\mathscr{A}$ and maps $\varepsilon^q\colon C^q\to I^{0,q}$ fitting into the diagram below satisfying the following:

(1) $I^{p,q}=0$ if either $p<0$ or $q<q_0$.
(2) The horizontal rows are exact, i.e., for each $q\geq q_0$, $C^q\to I^{\bullet,q}$ is an injective resolution.

(3) Let $Z^{p,q} = Z^{p,q}(I)$ be the kernel of the "vertical differential" $I^{p,q} \to I^{p,q+1}$. Then $Z^{\bullet,q}$ is an injective resolution of $Z^q = Z^q(C^\bullet)$, where $Z^q \to Z^{\bullet,q}$ is the natural map induced by $C^\bullet \to I^\bullet$.
(4) Let $B^{p,q} = B^{p,q}(I)$ be the image of the vertical differential $I^{p,q} \to I^{p,q+1}$. Then $B^{\bullet,q}$ is an injective resolution of $B^q = B^q(C^\bullet)$. The map $B^q \to B^{0,q}$ is (again) the natural map arising from $C^\bullet \to I^\bullet$.
(5) Let $H^{p,q} = H^{p,q}(I)$ be the q-th cohomology of the complex $I^{p,\bullet}$. Then $H^{\bullet,q}$ is an injective resolution of $H^q(C^\bullet)$ (again via $C^\bullet \to I^\bullet$).

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & C^{q+1} \arrow[r, "\varepsilon^{q+1}"] \arrow[u] & I^{0,q+1} \arrow[r] \arrow[u] & I^{1,q+1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & C^{q} \arrow[r, "\varepsilon^{q}"] \arrow[u] & I^{0,q} \arrow[r] \arrow[u] & I^{1,q} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & C^{q-1} \arrow[r, "\varepsilon^{q-1}"] \arrow[u] & I^{0,q-1} \arrow[r] \arrow[u] & I^{1,q-1} \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Such a "resolution" of $C^\bullet$ always exists. It is by no means unique. It is called a *Cartan-Eilenberg resolution* of $C^\bullet$. For completeness we provide a proof of its existence, though the reader is urged to skip the construction on the first reading.

Pick arbitrary injective resolutions for B^q and for $\mathrm{H}^p(C^\bullet)$, with the caveat that injective resolutions of zero objects will be chosen to be the zero injective resolution. Call these resolutions $B^{\bullet,q}$ and $H^{\bullet,q}$ respectively. Since

$$(*) \qquad 0 \to B^q \to Z^q \to H^q(C^\bullet) \to 0$$

is a short exact sequence of objects, one can use the *Horseshoe Lemma* to get an injective resolution of $Z^{\bullet,q}$ of Z^q which fits into a short exact sequence of complexes

$$(\dagger) \qquad 0 \to B^{\bullet,q} \to Z^{\bullet,q} \to H^{\bullet,q} \to 0$$

lifting $(*)$. Next we have an exact sequence

$$(**) \qquad 0 \to Z^q \to C^q \to B^{q+1} \to 0.$$

Since we have injective resolutions for the two ends of the short exact sequence, another application of the Horseshoe Lemma gives us an injective resolution $I^{\bullet,q}$ which fits into a short exact sequence of complexes

$$(\ddagger) \qquad 0 \to Z^{\bullet,q} \to I^{\bullet,q} \to B^{\bullet,q+1} \to 0$$

lifting $(**)$. Note that since we dealing with injective modules in $(\dagger)$ and $(\ddagger)$ we have decompositions.

$$(4.1.1) \qquad I^{p,q} = Z^{p,q} \oplus B^{p,q+1} = B^{p,q} \oplus H^{p,q} \oplus B^{p,q+1}.$$

It follows that for a fixed p the composite

$$I^{p,q} \twoheadrightarrow B^{p,q+1} \hookrightarrow I^{p,q+1} \tag{4.1.2}$$

gives a complex $I^{p,\bullet}$. In fact, as is easily checked, $I^{\bullet,\bullet}$ forms a double-complex and the notations we have used in the construction are consistent with the notations used in the list of requirements from a Cartan-Eilenberg resolution of $C^\bullet$.

The following (easy) Lemma is what gives us the Grothendieck spectral sequence.

Lemma 4.1.3. *Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive functor. Then for every pair of integers $(p,\, q)$ we have*

$$F(H^{p,q}) = \mathrm{H}^q(F(I^{p,\bullet})).$$

Proof. Since F is additive, it respects direct sums. Apply F to the decompositions in (4.1.1) to obtain

$$F(I^{p,q}) = F(B^{p,q}) \oplus F(H^{p,q}) \oplus F(B^{p,q+1}).$$

The q-th coboundary map for the complex $F(I^{p,\bullet})$ can be computed via (4.1.2), and it is the projection $F(I^{p,q}) \twoheadrightarrow F(B^{p,q+1})$ followed by the inclusion $F(B^{p,q+1}) \hookrightarrow F(I^{p,q+1})$. From here the q-cocycles and the q-coboundaries in the complex $F(I^{p,\bullet})$ are easily seen to be $F(B^{p,q}) \oplus F(H^{p,q})$ and $F(B^{p,q})$ respectively, giving the lemma. $\square$

4.2. **The Grothendieck spectral sequence.** Suppose $G\colon \mathscr{A} \to \mathscr{B}$ and $F\colon \mathscr{B} \to \mathscr{C}$ are covariant exact functors such that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives, and $G(I)$ is F-acyclic for injective objects I of $\mathscr{A}$. Recall that an object $B \in \mathscr{B}$ is called F-acyclic if $R^iFB = 0$ for $i \geq 1$. The Grothendieck spectral sequence for an object $X \in \mathscr{A}$ is one whose $E_2^{p,q}$ term is $\mathrm{R}^pF \circ \mathrm{R}^qG(X)$ and which abuts to $\mathrm{R}^n(F \circ G)(X)$. In other words

$$\mathrm{R}^pF \circ \mathrm{R}^qG(X)\mathbf{R} \Longrightarrow \mathrm{R}^n(F \circ G)(X). \tag{4.2.1}$$

Let $X \in \mathscr{A}$ and let $X \to J^\bullet$ be an injective resolution of X. Let

$$C^\bullet = G(J^\bullet).$$

Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $C^\bullet$. Let

$$K^{\bullet,\bullet} = F(I^{\bullet,\bullet})$$

and set $K^\bullet$ equal to the total complex of $K^{\bullet,\bullet}$. The map $C^\bullet \to I^\bullet$ induces a map

$$(F \circ G)(J^\bullet) = F(C^\bullet) \to F(I^\bullet) = K^\bullet.$$

As in the early parts of these notes, let $\{E_r^{p,q}\}$ denote the spectral sequence associated with the double-complex $K^{\bullet,\bullet}$. By Lemma 4.1.3 we see that $E_1^{p,q} = F(H^{q,p})$. Note that $H^{q,p}$ is an injective resolution of $\mathrm{H}^q(C^\bullet) = \mathrm{R}^qG(X)$. Therefore

$$E_2^{p,q} = (\mathrm{R}^pF \circ \mathrm{R}^qG)(X).$$

To show (4.2.1) we have to show that $\mathrm{R}^n(F \circ G)(X) \xrightarrow{\sim} \mathrm{H}^n(K^\bullet)$. Now, by hypothesis $G(J^i)$ is F-acyclic for each i since J^i is injective. Therefore the q-th row of $K^{\bullet,\bullet}$ is a resolution of the $FG(J^q)$. It follows that the map $(F \circ G)(J^\bullet) \to K^\bullet$ is a quasi-isomorphism, giving $\mathrm{R}^n(F \circ G)(X) \xrightarrow{\sim} \mathrm{H}^n(K^\bullet)$.

4.3. **The Leray Spectral Sequence.** This is a special case of the Grothendieck spectral sequence. For a topological space X, let Sh_X denote the abelian category of sheaves of abelian groups on X. If $g\colon X \to Y$ is a continuous map, then we have the direct image functor $g_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_Y$. Moreover for a flasque sheaf $\mathscr{F} \in \mathrm{Sh}_X$, the direct image $g_*(\mathscr{F}) \in \mathrm{Sh}_Y$ is clearly flasque. In particular, if $\mathscr{F}$ is injective $g_*(\mathscr{F})$ is flasque, since injectives are flasque. (Actually, $g_*(\mathscr{F})$ is also injective if $\mathscr{F}$ is, and this is because $g^{-1}\colon \mathrm{Sh}_Y \to \mathrm{Sh}_X$ is exact, $\mathrm{Hom}_Y(-,\, g_*(\mathscr{F})) \xrightarrow{\sim} \mathrm{Hom}_X(g^{-1}(-),\, \mathscr{F})$, and the last functor is exact.) Suppose $X \xrightarrow{g} Y \xrightarrow{f} Z$ is a pair of continuous maps between topological spaces. Since g_* sends injectives to flasques (in fact to injectives), and flasque sheaves are f_*-acyclic, therefore as a special case of the Grothendieck spectral sequence we have for any $\mathscr{F} \in \mathrm{Sh}_X$ the associated *Leray spectral sequence*:

$$(\mathrm{R}^p f_* \circ \mathrm{R}^q g_*)(\mathscr{F}) \Longrightarrow \mathrm{R}^{p+q}(f\circ g)_*\mathscr{F}. \tag{4.3.1}$$

In the event $Z = \{\star\}$ is a point, the above reduces to

$$\mathrm{H}^p(Y,\, \mathrm{R}^q g_*\mathscr{F}) \Longrightarrow \mathrm{H}^{p+q}(X,\, \mathscr{F}). \tag{4.3.2}$$

References

[EGA-01] A Grothendieck and J. Dieudonne, *Élements de Géométrie Algébrique*, Springer-Verlag, Heidelberg, 1971.

[EGA-III] A Grothendieck and J. Dieudonne, *Élements de Géométrie Algébrique* **III**, Publications Math. IHES **11**, Paris, 1961

[H] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.

[L1] J. Lipman, *Dualizing sheaves, differentials and residues on algebraic varieties*, Asterisque **117**, Société Mathématique de France, 1984.

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