LEFT ADJOINTS TO DIRECT IMAGES

PRAMATHANATH SASTRY

Everything is noetherian (schemes, rings etc) and separated for simplicity (things work more generally - but). For a scheme Z, Z_{qc} is the category of quasi-coherent $\mathscr{O}_Z$-modules. Suppose $f\colon X \to Y$ is affine. Let Y_{fqc} denote the category of quasi-coherent $f_*\mathscr{O}_X$-modules. Show that

$$f_*\colon X_{qc} \to Y_{fqc}$$

is an equivalence of categories. Note that the statement is trivially true when Y (and hence X) is affine, for then we are reduced to rings and modules. Cover Y by open affines, and note that the inverse images of these affines in X are affine and an open cover of X. Use an easy glueing argument to move from the results for the case $Y = \operatorname{Spec} A$ to the general case.

In particular, since $\mathscr{H}om_Y(f_*\mathscr{O}_X,\, \mathscr{G})$ is a quasi-coherent $f_*\mathscr{O}_X$-module for $\mathscr{G} \in Y_{qc}$, we have a unique object $f^!\mathscr{G} \in X_{qc}$ such that $f_*f^!\mathscr{G} = \mathscr{H}om_Y(f_*\mathscr{O}_X,\, \mathscr{G})$.

Now suppose $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, M a B-module and N an A-module. B is an A-algebra via f. The (Hom, $\otimes$)-adjointness gives us an isomorphism

$$(*) \qquad \operatorname{Hom}_B(M,\, \operatorname{Hom}_A(B,\, N)) \xrightarrow{\ \sim\ } \operatorname{Hom}_A(M,\, N).$$

Moreover one has a map $e\colon \operatorname{Hom}_A(B,\, N) \to N$ given by evaluation at 1, and $(*)$ is $\varphi \mapsto e \circ \varphi$.

Globalise these to the more general situation we are in (i.e., Y is not necessarily an affine scheme) to get

$$\operatorname{Hom}_X(\mathscr{F},\, f^!\mathscr{G}) \xrightarrow{\ \sim\ } \operatorname{Hom}_Y(f_*\mathscr{F},\, \mathscr{G}).$$

for $\mathscr{F} \in X_{qc}$ and $\mathscr{G} \in Y_{qc}$. One can ask, how well does this localise over Zariski open sets in Y? If f is *finite* and $\mathscr{F}$ is *coherent* then $f_{\mathscr{F}}$ is also coherent, and the isomorphism above localises well. Convince yourself that in this case we have the following generalisation of $(*)$

$$(**) \qquad f_*\mathscr{H}om_X(\mathscr{F},\, f^!\mathscr{G}) \xrightarrow{\ \sim\ } \mathscr{H}om_Y(f_*\mathscr{F},\, \mathscr{G}).$$

In other words convince yourself that when $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ and $A \to B$ is finite map, then $(**)$ is $(*)$. Finally, let $\tau_f(\mathscr{G})\colon f_*f^!\mathscr{G} = \mathscr{H}om_Y(f_*\mathscr{O}_X,\, \mathscr{G}) \to \mathscr{G}$ be the map which is locally the "e" (evaluation at 1) above. Show that on taking global sections, $\Gamma(X,\, (**))$ has the following description. Let $\varphi\colon \mathscr{F} \to f^!\mathscr{G}$ be a map in X_{qc} with $\mathscr{F}$ coherent. Then its image under $\Gamma(X,\, (**))$ is the composite

$$f_*\mathscr{F} \xrightarrow{f_*\varphi} f_*f^!\mathscr{G} = \mathscr{H}om_Y(f_*\mathscr{O}_X,\, \mathscr{G}) \xrightarrow{\tau_f} \mathscr{G}.$$

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References

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