

## DOUBLE COMPLEXES

We work throughout in an abelian category  $\mathcal{A}$  in which *countable direct sums* exist.

There are two related notions of a double complex. We give both versions below. The second (i.e. what we call an *anti-commuting* double complex below) is what you will often find in the older literature—and amongst non-algebraic-geometers. The first version (which we simply call a double complex, or sometimes a *standard double complex*) is the version given by Grothendieck in EGA, and is what most algebraic geometers and commutative algebraists are used to. The difference is one of convention.

**Standard Double Complexes.** A *double complex* in  $\mathcal{A}$ , or sometimes in our class, a *standard double complex* in  $\mathcal{A}$ , consists of data  $A^{\bullet\bullet} = (A, \partial_1, \partial_2)$ , where

$$A = (A^{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$$

is a family of objects in  $\mathcal{A}$ , and

$$\partial_1 = (\partial_1^{p,q})_{(p,q) \in \mathbb{Z}} \quad \partial_2 = (\partial_2^{p,q})_{(p,q) \in \mathbb{Z}}$$

are two families of morphisms

$$\partial_1^{p,q}: A^{p,q} \rightarrow A^{p+1,q} \quad \partial_2^{p,q}: A^{p,q} \rightarrow A^{p,q+1}$$

such that

$$\partial_1 \partial_1 = 0 \quad \partial_2 \partial_2 = 0 \quad \partial_1 \partial_2 = \partial_2 \partial_1.$$

We often suppress the superscripts  $p, q$  when these are either immaterial or easily deducible from the context. Thus, e.g., we write  $\partial_2$  for  $\partial_2^{p,q}$ . The maps  $\partial_1$  and  $\partial_2$  will be called partial coboundaries, and when we wish to be more specific, they will be called *horizontal* and *vertical* (partial) coboundaries respectively. The data fits into a commutative diagram, whose rows and columns are complexes.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 \\
 \dots & \xrightarrow{\partial_1} & A^{0,q+1} & \xrightarrow{\partial_1} & \dots & \xrightarrow{\partial_1} & A^{p,q+1} & \xrightarrow{\partial_1} & A^{p+1,q+1} & \xrightarrow{\partial_1} & \dots \\
 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \\
 \dots & \xrightarrow{\partial_1} & A^{0,q} & \xrightarrow{\partial_1} & \dots & \xrightarrow{\partial_1} & A^{p,q} & \xrightarrow{\partial_1} & A^{p+1,q} & \xrightarrow{\partial_1} & \dots \\
 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \uparrow \partial_2 & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & 
 \end{array}$$

Next consider the direct sum<sup>1</sup>

$$\mathrm{Tot}^n A^{\bullet\bullet} := \bigoplus_{p+q=n} A^{p,q}.$$

Define

$$\partial^n: \mathrm{Tot}^n A^{\bullet\bullet} \rightarrow \mathrm{Tot}^{n+1} A^{\bullet\bullet}$$

by the formula

$$\partial^n = \sum_{p+q=n} \{\partial_1^{p,q} + (-1)^p \partial_2^{p,q}\}.$$

The map within “curly brackets” can be regarded as a map  $A^{p,q} \rightarrow \mathrm{Tot}^{n+1} A^{\bullet\bullet}$ , taking values in the subobject  $A^{p+1,q} \oplus A^{p,q+1}$  of  $\mathrm{Tot}^{n+1} A^{\bullet\bullet}$ , whence by the definition of direct sum, the map  $\partial^n$  makes sense.

Evidently

$$\partial^{n+1} \circ \partial^n = 0$$

for every  $n \in \mathbb{Z}$  by the relations given between  $\partial_1$  and  $\partial_2$ . We have therefore a complex  $(\mathrm{Tot}^\bullet A^{\bullet\bullet}, \partial)$ , called the *total complex* associated to the double complex  $A^{\bullet\bullet}$ .

A morphism of double complexes  $\varphi: A^{\bullet\bullet} \rightarrow B^{\bullet\bullet}$  is (of course) a family of maps  $f^{p,q}: A^{p,q} \rightarrow B^{p,q}$ , one for each ordered pair of integers  $(p, q)$ , which commute with vertical and horizontal coboundaries. This naturally induces a map of complexes  $\mathrm{Tot} f: \mathrm{Tot}^\bullet A^{\bullet\bullet} \rightarrow \mathrm{Tot}^\bullet B^{\bullet\bullet}$

**Anti-commutative double complexes.** In much of the pre-Grothendieck literature, double complexes mean a variant of our standard double complexes. The only difference is that the grids in the diagram on the last page anti-commute rather than commute. In greater detail, for this course, data of the form  $K^{\bullet\bullet} = (K, d_1, d_2)$  represents an *anti-commuting* double complex if  $K$  is a family  $(K^{p,q})$  of objects in  $\mathcal{A}$  indexed by  $\mathbb{Z} \times \mathbb{Z}$  and  $d_1 = (d_1^{p,q}: K^{p,q} \rightarrow K^{p+1,q})$ ,  $d_2 = (d_2^{p,q}: K^{p,q} \rightarrow K^{p,q+1})$  are families of maps indexed by  $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ , called the *horizontal* and *vertical* partial coboundaries respectively, such that

$$d_1 d_1 = 0 \quad d_2 d_2 = 0, \quad d_1 d_2 = -d_2 d_1.$$

We set (and please pay attention to the notation, especially the accent on the top left)

$${}'\mathrm{Tot}^n K^{\bullet\bullet} := \bigoplus_{p+q=n} K^{p,q}$$

and define

$$d^n: {}'\mathrm{Tot}^n K^{\bullet\bullet} \rightarrow {}'\mathrm{Tot}^{n+1} K^{\bullet\bullet}$$

by the formula

$$d^n = \sum_{p+q=n} (d_1^{p,q} + d_2^{p,q})$$

without any sign of the form  $(-1)^p$  intervening. It is easy to see, with  $d := (d^n)_{n \in \mathbb{Z}}$ , that  $({}'\mathrm{Tot}^\bullet K^{\bullet\bullet}, d)$  is a complex. We call this complex the total complex associated with the anti-commuting double complex  $K^{\bullet\bullet}$ .

I will leave the task of defining maps of anti-commuting double complexes to you.

<sup>1</sup>This is where our assumption that  $\mathcal{A}$  has countable direct sums comes into play. Alternately, one can assume that the displayed direct sum for  $\mathrm{Tot}^n A^{\bullet\bullet}$  is finite for every  $n \in \mathbb{Z}$ .

**Bounded double complexes.** Let  $C^{\bullet\bullet}$  be a double complex (standard or anti-commutative). We say it is *bounded on the left* if there is an integer  $p_0$  such that

$$C^{p,q} = 0, \quad p < p_0.$$

If this happens we sometimes say  $C^{\bullet\bullet}$  is *bounded on the left by  $p_0$* . Similarly  $C^{\bullet\bullet}$  is *bounded below (by  $q_0$ )* if there exists an integer  $q_0$  such that

$$C^{p,q} = 0 \quad q < q_0.$$

I leave to you the fun task of defining terms like *bounded on the right* and *bounded above*.

Note that if  $C^{\bullet\bullet}$  is bounded on the left and below (resp. above and to the right) it lives in a translate of the first quadrant (resp. third quadrant) and as such the direct sum

$$\bigoplus_{p+q=n} C^{p,q}$$

is actually a finite sum<sup>2</sup> for each  $n$ . So in such instances, one can define  $\text{Tot}^n C^{\bullet\bullet}$  or  ${}^{\prime}\text{Tot}^n C^{\bullet\bullet}$  (as the case may be) without insisting that  $\mathcal{A}$  have countable direct sums. In fact we will largely be dealing with such situations.

**Tensor product of complexes and the Eilenberg-Zilber map.** Suppose  $A$  is a ring,  $M^\bullet$  a complex of right  $A$ -modules, and  $N^\bullet$  a complex of left  $A$ -modules. One has a double complex of abelian groups  $C^{\bullet\bullet}$  given by  $C^{p,q} = M^p \otimes_A N^q$  with obvious horizontal and vertical coboundary maps. In a somewhat confusing convention, the total complex  $\text{Tot}(C^{\bullet\bullet})$  is denoted  $M^\bullet \otimes_A N^\bullet$ . It should be denoted  $\text{Tot}(M^\bullet \otimes_A N^\bullet)$ , and some authors of late do do so. We will stick with the old convention for now. For the record

$$M^\bullet \otimes_A N^\bullet := \text{Tot}(C^{\bullet\bullet}).$$

Suppose now that  $x \in H^i(M^\bullet)$  and  $y \in H^j(N^\bullet)$  are two elements. Let  $\xi \in M^i$  and  $\zeta \in N^j$  be cocycles representing these cohomology classes. It is easy to see that  $\xi \otimes \zeta \in M^i \otimes_A N^j$  is an  $(i+j)$ -cocycle in  $M^\bullet \otimes_A N^\bullet$ . Moreover, one checks easily, the cohomology class of the cocycle  $\xi \otimes \zeta$  depends only on  $x$  and  $y$  and not on the representatives  $\xi$  and  $\zeta$ . I leave it to you work this out. The upshot is that we have a map of abelian groups:

$$H^i(M^\bullet) \otimes_A H^j(N^\bullet) \longrightarrow H^{i+j}(M^\bullet \otimes_A N^\bullet)$$

with  $x \otimes y$  mapping to the cohomology class of  $\xi \otimes \zeta$  where  $\xi$  and  $\zeta$  are as above. We therefore have (for each  $n \in \mathbb{Z}$ ), a map, the so called *Eilenberg-Zilber map*:

$$\mathcal{EZ}_n = \mathcal{EZ}_n(M^\bullet, N^\bullet): \bigoplus_{i+j=n} H^i(M^\bullet) \otimes_A H^j(N^\bullet) \longrightarrow H^n(M^\bullet \otimes_A N^\bullet).$$

It is straightforward to see that  $\mathcal{EZ}_n(M^\bullet, N^\bullet)$  is functorial in  $M^\bullet$  and in  $N^\bullet$ , i.e. it is *bifunctorial* in  $(M^\bullet, N^\bullet)$ .

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<sup>2</sup>Draw a picture with such quadrant translates, and look at the intersection of such quadrant translates with lines having slope  $-1$ .