THE WEYL CHARACTER FORMULA–II

PRAMATHANATH SASTRY

1. Introduction

These notes are short on proofs. The basic notions required to state the character formula are given. Proofs are pretty much the same as proofs for case of the unitary group $U(n)$ (see [2]), which is why they have been, in the main, omitted. Sadly, I have no time to say anything about the Peter-Weyl theorem, though I had promised something on it in [2]. It is easier to talk about semi-simple groups, and that is what we will do, for the modifications required to make things work for more general groups may well take us outside the realm of "basic notions". Ironically, $U(n)$, which we have taken as the model case, is not semi-simple.

In what follows G is a compact connected Lie group. We will specialize to semi-simple groups later.

2. Weyl Group

A subgroup T of G is called a *maximal torus* if T is a torus and there is no other torus in G containing T is a Lie subgroup. It turns out that all maximal tori are conjugate. This means that if $W(T)$ is the group $N(T)/T$, where T is a maximal torus, and $N(T)$ its normalizer in G, then different T yield isomorphic $W(T)$. The group $W(T)$ is called the *Weyl group of* (G, T). From now onwards we fix the maximal torus T, and denote $W(T)$ by W, and $N(T)$ by N.

Now, N acts on T. Indeed, each n yields the automorphism $t \mapsto ntn^{-1}$. However, the automorphism group of T is $GL(l, \mathbb{Z})$ where $l = \dim T$, and hence is *discrete*. Let N_o be the connected component of the identity of N. Our observations above show that N_o acts trivially on T. Consequently, if H is the image of a one parameter group in N_o, then $H \cdot T$ is a connected abelian group containing T. Maximality of T forces the equality $H \cdot T = T$. But such H generate N_o, and hence $N_o = T$. It follows (since N is compact) that N/N_o is *finite*, i.e. the Weyl group W is finite.

Note that since N_o acts trivially on T, we have an action of W on T.

3. The Conjugation Theorem

In this section we state the analogue of Lemma 3.1 of [2]. However, a few orienting remarks about Lie algebras are probably necessary. Recall that the The tangent space $\mathfrak{g}$ of G at the identity 1 can be identified with (left) invariant vector fields[1]. Elementary differential geometry tells us that if $\tilde{X}$ and $\tilde{Y}$ are two vector fields on G (thought of as $\mathbb{R}$—derivations of $C^\infty(G)$) then so is $[\tilde{X}, \tilde{Y}] = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}$. This gives a (non-associative) product on $\mathfrak{g}$, the so called *Lie bracket* which makes $\mathfrak{g}$ into

Date: October 21, 1999.

[1] i.e. vectoe fields $\tilde{X}$ on G such that ${l_g}_*\tilde{X} = \tilde{X}$ for every $g \in G$. Here l_g stands for left translation by g.

a *Lie algebra*[2] Moreover, the invariant vector fields are integrable, and the integral curves are complete (i.e. they are defined on all of $\mathbb{R}$). Let γ be the integral curve corresponding to $X \in \mathfrak{g}$, which satisfies $\gamma(0) = 1$. The *exponential of* X, written $\exp X$, is the element $\gamma(1) \in G$. This gives us the *exponential map*

$$\exp : \mathfrak{g} \longrightarrow G.$$

If G is abelian, then exp is a map of Lie groups ($\mathfrak{g}$ being identified with $\mathbb{R}^{\dim G}$) and in fact exp in this case is a covering map—thus $\exp : \mathfrak{g} \to G$ is the *universal covering space* of G.

For each $g \in G$, we have an inner automorphism $x \mapsto gxg^{-1}$ of G whence a (Lie-algebra) automorphism of $\mathfrak{g}$, which we denote $\mathrm{Ad}(g)$. This gives a Lie-group map

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

called the *adjoint representation* of G. It follows that we have a Lie algebra map

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$$

—the adjoint representation of the Lie algebra $\mathfrak{g}$. One checks easily that $\mathrm{ad}(X) = [X, \_\_]$.

For $x \in G$, the *centralizer* $\mathfrak{g}_x$ of x in $\mathfrak{g}$, is the subalgebra of $\mathfrak{g}$ of elements fixed by $\mathrm{Ad}(x)$. Fix a maximal torus T of G. If $\dim(T) = l$ and one can show $\dim \mathfrak{g}_x \geq l$ for all $x \in G$, and the set G' of x for which equality holds is open and dense. If x lies in G', it is called *regular* (cf. [2], bottom of p.2). The integer l is called the *rank* of G. Let T' be the regular elements in T.

We are now in a position to state the conjugacy theorem (cf. [2], Lemma 3.1. See [1], p.161, Prop. 1.8 for a proof). Let

$$D : T \longrightarrow \mathbb{R}$$

be the map

$$t \mapsto \det \left[\mathrm{Ad}(t^{-1}) - 1\right]_{\mathfrak{g}/\mathfrak{t}}$$

where $\mathfrak{t}$ is the Lie algebra of T[3].

Theorem 3.1. *For $x \in G$ and $t \in T$, the element xtx^{-1} depends only on the coset xT. If $\Psi(xT, t) = xtx^{-1}$,*

$$\Psi : G/T \times T' \to G'$$

is an analytic map making $G/T \times T'$ into a (possibly disconnected) covering space with covering degree equal to $|W|$. Moreover if dx, dt, $d\bar{x}$ are the normalized invariant measures on G, T and $\bar{G} = G/T$, then

$$\Psi^*(dx) = D d\bar{x} dt.$$

Before leaving this section we make one last comment. It turns out that the conjugation theorem above can be used to prove that two maximal tori are conjugate to each other (see, for e.g., [1], pp.159–160).

[2]In other words, (a)$[X, Y] = -[Y, X]$ and (b)$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, for $X, Y, Z \in \mathfrak{g}$.

[3]We may regard $\mathfrak{t}$ as a subalgbra of $\mathfrak{g}$ in a canonical way

4. Lessons from the $U(n)$ case

W acts on T and hence on $L^2(T)$. For $f \in L^2(T)$, and $w \in W$, f^w will denote the action of w on f. For $\chi \in \widehat{T}$, set

$$\widetilde{\chi} = \sum_{w\in W} \epsilon(w)\chi^w.$$

Call $\chi \in \widehat{T}$ *regular* if the χ^w are distinct for distinct $w \in W$ [4]. From the lessons learnt in proving Weyl's character formula for $U(n)$ we would like to do the following:

- Identify a "nice" subset P of $\widehat{T}$, such that $\{\widetilde{\chi}\,|\,\chi \in P\}$ forms a basis for the additive subgroup $\mathcal{F}$ of $L^2(T)$ consisting of functions Φ with *finite Fourier series* having *integral* coefficients, such that Φ is *skew symmetric*, i.e. $\Phi^w = \epsilon(w)\chi^w$[5]. (See Lemma 4.1 of [2]). Note that $\widetilde{\chi} = 0$ if χ is *not* regular, and hence P must necessarily consist of regular elements. If χ_1 and χ_2 are in the same W orbit, then $\widetilde{\chi_1}$ and $\widetilde{\chi_2}$ differ by a sign and hence both cannot be in P. If on the other hand χ_1 and χ_2 are in *different* W orbits, then $\widetilde{\chi_1}$ and $\widetilde{\chi_2}$ are clearly orthogonal, and hence P provides us with an orthogonal basis of the vector space $V = \mathcal{F} \otimes \mathbb{C}$. Note that $||\widetilde{\chi}||^2 = |W|$ for regular χ. (Cf. [2] p.5, Lemma 4.1).
- Let
$$D(t) = \det\left[\mathrm{Ad}(t^{-1} - 1\right]_{\mathfrak{g}/\mathfrak{t}} \qquad (t \in T).$$
We would like to have a factorization
$$D = \Delta\bar{\Delta}$$
where $\Delta \in \mathcal{F}$. Moreover, we would like $\Delta = \widetilde{\chi}_\rho$ for some $\rho \in P$ (cf. the proof of [2], p.6, Thm. 4.1).

Given the above, then as in [2] Theorem 4.1, we can prove that there is a on-one correspondence between $\widehat{G}$ and P such that for $\chi \in P$, if Θ_χ is the character of the corresponding irreducible representation of G, then

$$(\Theta_\chi)_T = \widetilde{\chi}/\Delta.$$

This then would be the character formula[6]. The requirements listed above put restrictions on the compact group G.

5. Positive characters and the Character formula

For a Lie algebra $\mathfrak{h}$ over $\mathbb{R}$, let $\mathfrak{h}_\mathbb{C}$ denote its *complexification*, i.e. $\mathfrak{h}_\mathbb{C} = \mathfrak{H} \otimes \mathbb{C}$. The map $\mathrm{ad}(\mathfrak{t}_\mathbb{C})$ acts on $\mathfrak{g}_\mathbb{C}$, and since T is comapct and abelian, this action is completely reducible and breaks up $\mathfrak{g}_\mathbb{C}$ into a direct sum

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$$

[4]For example, if $G = U(n)$, χ_m is regular if and only if $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ has distinct coordinates.

[5]For $G = U(n)$, P was the set $\{(m_1, \ldots, m_n)\,|\, m_1 > \ldots > m_n\}$.

[6]A little thought shows that much is required of the "nice" set P. The properties we have listed will only give the ambiguous formula $(\Theta_\chi)_T = \pm\widetilde{\chi}/\Delta$.

each summand being non-zero and stable under $\mathfrak{t}_{\mathbb{C}}$. Moreover, $\mathfrak{t}_{\mathbb{C}}$ acts on $\mathfrak{g}_\alpha$ via linear functional on $\mathfrak{t}_{\mathbb{C}}$, which we also denote by α. In other words

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} \,|\, [H,\, X] = \alpha(H)X,\, H \in \mathfrak{t}_{\mathbb{C}}\}.$$

The set R is called the set of *roots*, and each $\mathfrak{g}_\alpha$ is called *root space*. They are all of dimension 1. Thinking of elements of R as functionals, it turns out that for $H \in \mathfrak{t}$, $\alpha(H) \in i\mathbb{R}$. Indeed, each root space is stable under $\mathrm{Ad}(T)$ which acts on $\mathfrak{g}_\alpha$ via an element $\widehat{T}$. This element is written ξ_α, and is called the *global root*. We then have

$$\mathrm{Ad}(t) \cdot X = \xi_\alpha(t) \cdot X \qquad\qquad (t \in T, X \in \mathfrak{g}_\alpha.$$

Now, for $H \in \mathfrak{t}$, $\xi_\alpha(\exp H) = e^{\alpha(H)}$. It follows that $\alpha(H)$ is purely imaginary.

One checks that $[\mathfrak{g}_\alpha,\, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in R$. If the sum of α and β is zero, then the above bracket lies in $\mathfrak{t}_{\mathbb{C}}$. Otherwise it is zero.

Remove from $i\mathfrak{t}$ the finite number of hyperplanes determined by the functionals $\alpha \in R$. We get a dense open set whose connected compontnents are called *Weyl chambers*. Each chamber is a convex cone. Fix one chamber, C^+. The subset R^+ of roots which are positive on C^+ are called *positive roots* with respect to C^+.

If G is *semi-simple*, i.e. if its fundamental group is finite, then one has a natural inner product on $\mathfrak{g}$ given by the *Cartan-Killing form* $(X,\, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)$. Consequently one can talk about reflections s_α along the hyperplane determines by $\alpha \in R$. The Weyl group can be interpreted as the group generated by s_α as α varies over the roots. Note that W acts on $i\mathfrak{t}$. In fact it acts simply transitively on the set of chambers.

Any $\chi \in \widehat{T}$ gives rise to an element $\lambda_\chi \in i\mathfrak{t}^*$ such that

$$\chi(\exp H) = e^{\lambda_\chi(H)}.$$

Indeed, any map from $T \to S^1$ gives rise to a map between the universal covering spaces, and this map is unique once base points are fixed. Taking $\mathfrak{t}$ and $i\mathbb{R}$ as the respective covering spaces of T and S^1, one can get λ_χ. In fact $\chi \mapsto \lambda_\chi$ embedds $\widehat{T}$ as a lattice in $i\mathfrak{t}^*$. Using the inner product on $\mathfrak{t}$ (induced from $\mathfrak{g}$) we can identify $i\mathfrak{t}^*$ with $i\mathfrak{t}$. Let $H_{\lambda_\chi} \in i\mathfrak{t}$ be the element corresponding to λ_χ. We identify the set $P \subset \widehat{T}$ with the set of characters χ such that H_{λ_χ} lies in C^+. A character χ is called *positive* if it lies in P.

It remains to define Δ. Assume that

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

is also a positive root. Groups G for which this is true are called *acceptable*. Simply connected G are always acceptable. Then set

$$\Delta = \xi_\rho^{-1} \prod_{\alpha \in R^+} (\xi_\alpha - 1)\,.$$

Note that Δ depends on the choice of the positive chamber C^+ and as such it is better to denote it Δ^+.

One checks that P and Δ have all the properties outlined in Section 4. The road to the character formula is now clear.

Since it is now 2:50 p.m., perhaps this is the place to stop!!

References

[1] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*. Springer-Verlag, New York, 1985.

[2] P. Sastry. Weyl character formula -I. Basic notions seminar, Nov. 11, 1997.

The Mehta Research Institute, Chhatnag, Jhusi, Allahabad District, U.P., INDIA 221 506

E-mail address: pramath@mri.ernet.in