THE WEYL CHARACTER FORMULA–I

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1. Introduction

This is the first of two lectures on the representations of compact Lie groups. In this lecture we concentrate on the group $U(n)$, and classify all its irreducible representations. The methods give the paradigm and set the template for studying representations of a large class of compact Lie groups—a class which includes semi-simple compact groups. The climax is the character formula of Weyl, which classifies all irreducible representations for this class, up to unitary equivalence. The typing was done in a hurry (last night and this morning, to be precise), and there may well be more serious errors than mere typos. Typos of course abound.

2. Fourier Analysis on abelian groups

It is well-known, and easy to prove, that a compact ableian Lie group [1] is necessarily a torus $\mathbb{T}^n = S^1 \times \ldots \times S^1$ (n factors). A *character* on $\mathbb{T} = \mathbb{T}^n$ is a Lie group map $\chi : \mathbb{T} \to \mathbb{C}^*$ [2]. The maps $\chi_{m_1,\ldots,m_n}$ $((t_1, \ldots, t_n) \mapsto t_1^{m_1} \ldots t_n^{m_n})$ lists all characters on $\mathbb{T}$ as $(m_1, \ldots, m_n)$ varies over $\mathbb{Z}^n$. This is a fairly elementary result—indeed, the image of a character χ must be a compact connected subgroup of $\mathbb{C}^*$, forcing χ to take values in $S^1 \subset \mathbb{C}^*$. From there to seeing χ must equal a $\chi_{m_1,\ldots,m_n}$ is easy (try the case $n = 1$!).

$\mathbb{T}$ has an *invariant* Borel measure $dt = dt_1 \ldots dt_n$ (the product of the "arc length" measures on the various S^1 factors), the so called *Haar measure* on $\mathbb{T}$ [3]. We denote $L^2(\mathbb{T}, dt)$ by $L^2(\mathbb{T})$. The classical result that "periodic" functions on $\mathbb{R}^n$ have an expansion by Fourier series can be reformulated in a more conceptual way—in fact as a special case of the Peter-Weyl theorem (see also 4.1).

Theorem 2.1. *There is a Hilbert space decomposition (i.e. an orthogonal decomposition)*

$$L^2(\mathbb{T}) = \bigoplus_{m \in \mathbb{Z}^n} \mathbb{C} \cdot \chi_{-m}$$

Date: October 21, 1999.

[1] Recall that a Lie group is a C^∞ (resp. (real) analytic) manifold which is also a group, such that the the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are C^∞ (resp. analytic). I will only consider *compact*, *analytic* Lie groups. The requirement of analyticity is no restriction in view of the classical result that every C^∞ Lie group has a unique real analytic structure on it compatible with its C^∞ structure.

[2] In general, for a compact group, a character is the trace function associated with a finite dimensional representation. What we have called a character, is really an *irreducible* character of $\mathbb{T}$.

[3] On compact groups we have a unique measure, the Haar measure, which is invariant under both right and left translations, and which gives the whole group measure one. If the group is in addition a Lie group, then the measure can be represented by an analytic k-form, where k is the dimension of the group.

In particular, if $(\ ,\ )$ *denotes the inner product on* $L^2(\mathbb{T})$*, then every* $f \in L^2(\mathbb{T})$ *has a Fourier series expansion*

$$f = \sum_m \hat{f}(m)\chi_{-m}$$

where

$$\hat{f}(m) = (f,\, \chi_{-m}).$$

Remark 2.1. For $f \in L^2(\mathbb{T})$, the theorem also gives the Parseval equality

$$||f||^2 = \sum_{m\in\mathbb{Z}} |\hat{f}(m)|^2.$$

Note that $\hat{f}$ can be thought of as a member of $L^2(\mathbb{Z}^n, \sharp)$, where $\sharp$ is the "counting measure" on $\mathbb{Z}^n$. Thought of this way, the Fourier transform $f \mapsto \hat{f}$ becomes, via Parseval's relation, a unitary isomorphism between the Hilbert spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{Z}^n, \sharp)$. Note that $\sharp$ is "the" Haar measure on $\mathbb{Z}^n$. These remarks are a special case of Pontrayagin duality, with $\mathbb{T}$ and $\mathbb{Z}^n$ being considered as abelian groups "dual" to each other.

3. The group $U(n)$

Recall that a *unitary matrix* is a square matrix A, with entries in $\mathbb{C}$, satisfying $A^*A = 1$ (here, as elsewhere, A^* is the Hilbert space adjoint, or, what amounts to the same thing, the conjugate transpose). Such an A is also characterized by the fact that it preseves the Hermitian inner product on $\mathbb{C}^n$ (assuming A to be $n \times n$). Let $U(n)$ denote the group of $n \times n$ matrices. One shows, in the usual way, that $U(n)$ is a compact subgroup of $GL(n,\, \mathbb{C})$[4] Consequently $U(n)$ is a compact Lie group, since closed subgroups of Lie groups are themselves Lie groups (see [1], p.28, Thm. (3.11).)

Let $D \subset U(n)$ be the Lie subgroup consisting of diagonal matrices, with diagonal entries of modulus 1. Then $D \cong \mathbb{T}^n$. We claim that given $g \in U(n)$, the *conjugacy class* C_g of g in $U(n)$ must intersect D. This is another way of saying that we can find an orthonormal basis of $\mathbb{C}^n$ consisting entirely of eigenvectors of g. This is clearly true for $n = 1$. Let V be an eigenspace of g, and W its orthogonal complement. Then $gW = W$, for g preserves inner products. Moreover g acts as a unitary operator on W. By induction on dimension, we may assume that W has an orthonormal basis of eigenvectors of g. The space V certainly has such a basis, and hence $\mathbb{C}^n$ has the required decomposition.

A little thought shows that the conjugacy class C_g meets D in a finite number of points—in fact in at most $n!$ points. If $|C_g \cap D| = n!$ we call g *regular* [5]. If an element of $U(n)$ is not regular, it is called *singular*.

Let G denote $U(n)$, G' the regular locus of G, and D' the regular elements in D. It turns out that G' is an open sense subset of G (and hence its complement in G has zero Haar measure).

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[4]The "usual way" involves showing that $U(n)$ is closed and bounded on $\mathbb{C}^{n^2}$, using the fact that norms on $\mathbb{C}^{n^2}$ are equivalent. Since the operator norm of any unitary matrix is 1, $U(n)$ is bounded. The property of preserving the inner product clearly carries over to limits, whence $U(n)$ is closed.

[5]In other words, the eigenvalues of g are distinct. There are other characterisations of regular elements of $U(n)$. The element g is regular if and only if C_g has maximal dimension. This characterization allows for generalization to other compact groups.

We are interested in characterizing the (finite dimensional) irreducible representations of $G = U(n)$. Our strategy will be to reduce it to representation theory over $D \cong \mathbb{T}^n$, where classical Fourier theory applies. One needs to relate integrals of *class functions* (i.e. functions which are constant on conjugacy classes) on G to integrals over D. The following two results are crucial to this strategy and set the pattern for similar results over arbitrary compact groups (see [2], pp. 26–28 and also [1], pp. 159–163).

Lemma 3.1. *For $x \in G$, $t \in D$, xtx^{-1} depends only on the coset xD. If $\Psi(xD, t) = xtx^{-1}$,*

$$\Psi : G/D \times D' \to G'$$

is an analytic map making $G/D \times D'$ into a (possibly disconnected) covering space over G' with covering degree equal to $n!$. Moreover, if

$$\Delta(t) = \prod_{1 \le i < j \le n} (t_i - t_j) \qquad (t = diag(t_1, \ldots, t_n))$$

and dx, dt, $d\bar{x}$ are the normalized invariant measures on G, D and $\bar{G} = G/D$, then

$$\Psi^*(dx) = \Delta\bar{\Delta} d\bar{x} dt.$$

Proof. Clearly $(xD, t) \mapsto xtx^{-1}$ is well defined. It is not hard to see that Ψ is analytic. We now compute $d\Psi$. To do this, one checks that one only needs to compute $d\Psi_{(\bar{1},t)}$ from the tangent space T_t at $(\bar{1}, t) \in G/D \times D'$ to the tangent space T'_t at $t \in G'$. Both these spaces (i.e. T_t and T'_t) can be identified (in a canonical way) with

$$\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{d}$$

where $\mathfrak{q}$ is the space of skew hermitian matrices with zeros in the diagonal, and $\mathfrak{d}$ is the space of diagonal matrices. We may then think of $d\Psi_{(\bar{1},t)}$ as an endomorphism of $\mathfrak{g}$. One checks that this endomorphism is

$$(Z, H) \mapsto (t^{-1}Zt - Z, H).$$

For $u \in D$, let $A(u)$ be the endomorphism

$$Z \mapsto u^{-1}Zu - Z$$

of $\mathfrak{q}$. A little thought shows that if $\Psi^*(dx) = \omega(\bar{x}, t)d\bar{x}dt$, then $\omega(\bar{x}, t) = \det A(t)$. Going over to the complexification $\mathfrak{q}_{\mathbb{C}}$ of $\mathfrak{q}$ (note that $\mathfrak{q}_{\mathbb{C}}$ consists of all complex matrices with diagonal entries zero), and denoting by E_{rs} $(r \neq s)$ the matrix $(\delta_{ir}\delta_{js})_{ij}$, we see that

$$A(t)E_{rs} = (\frac{t_s}{t_r} - 1)E_{rs}.$$

It follows that

$$\begin{aligned} \det A(t) &= \prod_{r \neq s} (\frac{t_r}{t_s} - 1) \\ &= \prod (\frac{t_r - t_s}{t_s})(\frac{\bar{t_r} - \bar{t_s}}{\bar{t_s}}) \\ &= \Delta\bar{\Delta} \qquad\qquad (\text{since } |t_s| = 1 \text{for all } s). \end{aligned}$$

It only remains to show that Ψ is a covering map with fibre cardinality $n!$. The statement about the fibre cardinality is clear. To see Ψ is a covering map, note that the map $(xD, t) \mapsto xtx^{-1}$ is a map between the two compact manifolds $G/D \times D$

and G, and hence is proper. The map Ψ is obtained by "base changing" to G' G', and hence Ψ is proper. But we have shown that Ψ is a local homeomorphism. It must be a covering map! $\square$

One immediately has (by first looking at continous f, and then using a standard argument) the following.

Theorem 3.1. *Let $f \in L^1(G)$. Then*

$$\int_G f dx = \frac{1}{n!}\int_D \int_G f(xtx^{-1})\Delta\bar{\Delta} dx dt.$$

In particular, if f is a "class function", and f_D its restriction to D, then

$$\int_G f dx = \frac{1}{n!}\int_D f_D \Delta\bar{\Delta} dt.$$

4. Weyl's Character Formula for $U(n)$

Before discussing the character formula for $U(n)$, I would like to make a few comments about compact Lie groups and their representations. The accepted rule of the thumb is the following. Proofs of results for compact groups are often the same as that of finite groups H, with the operator $\int_G (\_\_) dg$ replacing the operator $1/|H| \sum_{h\in H}$. Note that $1/|H| \sum_{h\in H} = \int_H (\_\_) dh$. This philosophy readily gives the following:

- All finite dimensional representations on G are equivalent to a unitary representation. From now onwars we only deal with finite dimensional unitary representations [6].
- Every (finite dimensional, unitary) representation breaks up into an orthogonal direct sum of irreducible representations.
- For π a representation of G, let Θ_π denote the associated character $\Theta_\pi : G \to \mathbb{C}$, $g \mapsto \mathrm{tr}\pi(g)$. Then $\Theta_\pi = \Theta_{\pi'}$ if and only if $\pi \cong \pi'$, where $\cong$ denotes unitary equivalence [7]. We denote the unitary equivalence class of π by $[\pi]$. Thus Θ_π can with equal justice be denoted $\Theta_{[\pi]}$, and we will often do this. Let
$$\hat{G} = \{[\pi] \,|\, \pi \text{irreducible}\}.$$
- Let (,) denote the inner product on $L^2(G)$. Then for $\omega, \omega' \in \hat{G}$,
$$(\Theta_\omega,\, \Theta_{\omega'}) = \delta_{\omega\omega'}.$$
- If $\pi = m_1\pi_1 + m_2\pi_2 + \ldots + m_r\pi_r$, each π_i irreducible and in distinct equivalence classes, then
$$||\Theta_\pi||^2 = m_1^2 + \ldots + m_r^2.$$
  In particular
$$||\Theta_\pi|| = 1 \Leftrightarrow [\pi] \in \hat{G}.$$

[6]Except for a brief while in the second lecture when we discuss the Peter-Weyl Theorem and relax the finite dimensional restriction. The representation space will be $L^2(G)$ and π will be the "regular" representation.

[7]In other words if V is the space of π and V' that of π', then there exists a unitary isomorphism $T : V \to V'$ (or a unitary "intertwining operator") such that $\pi'(g)\circ T = T\circ\pi(g)$, $g \in G$.

Remark 4.1. While the above results have an easy passage from the finite groups to compact groups, not all proofs of results about finite group representations lend themselves to a mechanical reworking in the compact situation— in particular, even producing a finite dimesnional representation of G requires the Peter-Weyl theorem. For the analysis of $U(n)$ we only need a part of the Peter-Weyl theorem, viz. the fact the irreducible characters form an orthonormal basis of the space of class functions in $L^2(G)$.

We now return to $U(n)$. In what follows G will once again be the group $U(n)$. The symmetric group S_n will be denoted by W. W is the "Weyl group" og $U(n)$. For any function Φ in $L^2(D)$, and $w \in W$, let Φ^w denote the action of w on Φ[8]. For a character $\chi \in \hat{D}$, let

$$\tilde{\chi} = \sum_{w \in W} \epsilon(w) \chi^w$$

where $\epsilon(w)$ is the sign of the permutation w. Recall that $\hat{D} \cong \mathbb{Z}^n$. Let $\chi_{m_1, \dots, m_n}$ denote the character associated to $(m_1, \dots, m_n) \in \mathbb{Z}^n$. Note that $\chi_m^w = \epsilon(w)\chi_{w(m)}$.

Lemma 4.1. *Let $\Phi \in L^2(D)$ be such that*

1. *Φ has a finite Fourier series expansion with integer coefficients;*
2. *Φ is skew symmetric, i.e.*

$$\Phi^w = \epsilon(w)\Phi \qquad\qquad (w \in W).$$

Then Φ can be written uniquely as

$$\Phi = \sum_{m_1 > \dots > m_n} c(m)\tilde{\chi_m}$$

where $c(m_1, \dots, m_n) \in \mathbb{Z}$. The $\tilde{\chi_m}$ occuring in the expansion are orthogonal, and

$$||\Phi||^2 = n! \sum |c(m)|^2.$$

Proof. Let $V \subset L^2(D)$ be the complex vector subspace spanned by $\tilde{\chi_m}$, $m \in \mathbb{Z}^n$. Let

$$B = \{\tilde{\chi_m} |\, m_1 > \dots > m_n\}.$$

Then B is an orthogonal basis of V. This can be seen by checking the following

- $\tilde{\chi}_{m_1, \dots, m_n} = \epsilon(w)\tilde{\chi}_{w(m_1), \dots, w(m_n)}$.
- $\tilde{\chi}_{m_1, \dots, m_n} = 0$ if the $m_1, \dots, m_n$ are not distinct (follows from above identity).
- If χ_1 and χ_2 belong to distinct W orbits, then $(\tilde{\chi}_1, \tilde{\chi}_2) = 0$.

Let Φ be as in the statement of the Lemma and let $\sum_m d(m)\chi_m$ be its Fourier expansion (the $d(m)$ are integers, and only a finite number of them are non zero). Then $n!\Phi = \sum_m d(m)\tilde{\chi}_m$. Hence Φ lies in V. If

$$\Phi = \sum c(m)\tilde{\chi_m}$$

is its expansion with respect to the basis B, then $c(m)$ is the coefficient of χ_m in the Fourier expansion of Φ. Hence, by hypotheses on Φ, $c(m)$ is an integer. The statement about the norm of Φ follows from the obvious fact that $||\chi||^2 = n!$ if $\chi \in B$. ∎

[8] W acts on D by permuting the diagonal elements, and hence there is a dual action of W on $L^2(D)$.

Theorem 4.1. *(Weyl's character formula) The irreducible characters of $U(n)$ are in natural one-one correspondence with n-tuples of integers $(m_1, \dots, m_n)$ with $m_1 > \dots > m_n$. For such an $(m_1, \dots, m_n)$ let $\Theta_{m_1, \dots, m_n}$ denote the corresponding character. Then*

$$(\Theta_{m_1, \dots, m_n})_D = \tilde{\chi}_{m_1, \dots, m_n}/\Delta.$$

Proof. Let π be an irreducible representation of $U(n)$, and let Θ_π be the corresponding character. Let

$$\Phi_\pi = (\Theta_\pi)_D \Delta.$$

Then Φ_π satisfies the hypotheses of Lemma 4.1. Let

$$\Phi_\pi = \sum c(m)\tilde{\chi}_m$$

be the expansion that Lemma 4.1 guarantees. Using Theorem 3.1, one sees that $||\Phi_\pi||^2 = n!$. Using Lemma 4.1 once again we see that

$$\sum |c(m)|^2 = 1.$$

But the $C(m)$'s are integers. It follows that

$$\Phi_\pi = c(m_1, \dots, m_n)\tilde{\chi}_{m_1, \dots, m_n}$$

for some $m_1 > \dots > m_n$, with $c(m_1, \dots, m_n) = \pm 1$. Thus, for each $[\pi] \in \hat{G}$, there are integers $m_i(\pi)$, with $m_1(\pi) > \dots > m_n(\pi)$. We have to show that $c(m_1(\pi), \dots, m_n(\pi)) = 1$ and that every $(m_1, \dots, m_n)$ with $m_1 > \dots m_n$ occurs as $(m_1(\pi), \dots, m_n(\pi))$ for some $[\pi] \in \hat{G}$. To see the latter, let $q_1 > \dots q_n$ be integers such that q cannot be written as $m(\pi)$. Then $\tilde{\chi}_q$ is not equal to any $\tilde{\chi}_{m(\pi)}$. It follows that the inner product $(\tilde{\chi}_q, \tilde{\chi}_{m(\pi)})$ vanishes. Let Θ_D be the W invariant function $\tilde{\chi}_q/\Delta$. Then Θ_D has a unique extension to a class function Θ on G (use Lemma 3.1). By Theorem 3.1, $(\Theta, \Theta_\pi) = 0$ for every $[\pi] \in \hat{G}$. This contradicts the completeness of the $\{\Theta_{[\pi]}\}$ amongst class functions.

It remains to show that $c(m_1(\pi), \dots, m_n(\pi)) = 1$. Give $\mathbb{Z}^n$ its lexicographic order. Write the Fourier expansion

$$(\Theta_\pi)_D = c\chi_{r_1, \dots, r_n} + \sum_{s<r} d(s)\chi_s.$$

Then $c, d(s)$ are all positive integers. Now,

$$\Delta = \tilde{\chi}_{(n-1, n-2, \dots, 0)}.$$

Hence

$$\Phi_\pi = c\chi_{r_1+n-1, r_2+n-2, \dots, r_n} + \dots$$

where the unwritten terms are those whose suffixes are less than $(r_1, \dots, r_n)$. Now the highest index in $c(m(\pi))\tilde{\chi}_{m(\pi)}$ is precisely $(m_1(\pi), \dots, m_n(\pi)$. Hence

$$c(m(\pi)) = c, r_i + n - i = m_i(\pi), i = 1, \dots, n.$$

Since $c > 0$, $c(m(\pi)) = 1$ and we are done! □

Remark 4.2. Note that in the above $r_1 \geq r_2 \geq \dots \geq r_n$. Further, the proof shows that $r_1, \dots, r_n$ determines $m_1, \dots, m_n$ via the formulae $m_i = r_i + n - i$. Stated another way, this says that if one knows the highest index in the Fourier expansion of $(\Theta_\pi)_D$, then one can recover π. This leads to:

Theorem 4.2. *(Cartan-Weyl description via highest weights). If* $r_1, \ldots, r_n$ *are integers* $r_1 \geq r_2, \ldots \geq r_n$ *there is (up to equivalence) a unique irreducible representation of* $U(n)$ *with highest weight* $\chi_{r_1, \ldots, r_n}$ *and this weight has multiplicity* 1 *in the representation. Its character is* $\Theta_{m_1, \ldots, m_n}$ *where* $m_i = r_i + n - i$ *and all irreducible representations are thus obtained.*

References

[1] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*. Springer-Verlag, New York, 1985.

[2] V. S. Varadarajan. *An Introduction to Harmonic Analysis on Semi simple Lie Groups*. Cambridge University Press, Cambridge, 1989.

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